

Topological K-theory and its Chern character for non-commutative spaces

Anthony Blanc

I3M, Université Montpellier 2

anthony.blanc@math.univ-montp2.fr

December 3, 2012

Abstract

The purpose of this work is to give a definition of a topological K-theory for dg-categories over \mathbb{C} and to prove that the Chern character map from algebraic K-theory to periodic cyclic homology descends naturally to this new invariant. This new map provides a natural candidate for the existence of a rational structure on the periodic cyclic homology of a smooth and proper dg-algebra.

The main ingredient in the definition of topological K-theory is the geometric realization functor for simplicial presheaves on the site of complex algebraic varieties. Our first main result states that the geometric realization of the presheaf of connective algebraic K-theory is **bu**, the connective complex topological K-theory spectrum. The same works for non-connective algebraic K-theory, but for this we are lead to prove a proper hyperdescent result. This enables us to define topological K-theory by inverting the Bott element. The fact that the Chern character descends to this invariant is then established by using the Künneth formula for periodic cyclic homology and the hyperdescent properties of the derived geometric realization functor.

Contents

1	Introduction	2
2	Preliminaries	6
2.1	Δ -spaces, Γ -spaces and connective spectra	6
2.2	Connective algebraic K-theory	10
2.3	Non-connective algebraic K-theory	11
2.4	Geometric realization over complex numbers	13
3	Algebraic Chern character	19
3.1	Cyclic homologies spectra	19
3.2	The algebraic Chern character as a K-linear map	20
4	Topological K-theory and its Chern character	22
4.1	Semi-topological and Topological K-theory	22
4.2	Connective theory, bu and the moduli stack of perfect dg-modules	23
4.3	Proof of Proposition 4.6	27
4.4	Non-connective theory and proper hyperdescent in topology	29
4.5	Topological Chern character	35

1 Introduction

Any algebraic variety gives rise to a dg-category, namely perfect complexes of quasi-coherent sheaves. Moreover this dg-category shares many of the finiteness properties of its variety, e.g. being smooth or proper. This is the non-commutative (or categorical) algebraic geometry point of view as exposed for example in the work of Kartzarkov–Kontsevich–Pantev [22], in which any dg-category (or A_∞ -category) is understood as the dg-category of complexes of sheaves on an hypothetic non-commutative space.

This point of view rises many questions about how to extend the mathematical structures we know in the context of algebraic varieties to the categorical framework. For example the De Rham cohomology of a smooth and proper complex algebraic variety carries a natural Hodge structure, whose rational part is given by the rational Betti cohomology. The natural categorical counterpart of De Rham cohomology is known (via the HKR theorem) to be periodic cyclic homology, while Hochschild homology of dg-categories is the analog of Hodge cohomology of varieties. Following this circle of ideas, several mathematicians (beginning with Kontsevich–Soibelman [25]) have expected the periodic cyclic homology $HP(A)$ of a smooth and proper dg-algebra A to carry a so called non commutative Hodge structure which generalizes the commutative case, see [22, Conjecture 2.24] for a precise version.

This hypothetical non-commutative Hodge structure can be basically described by two sets of data : the De Rham data and the Betti data. The first is the analog of the Hodge filtration and the second is the analog of the rational structure. The famous degeneration conjecture ([22, 2.2.4]) furnishes a sufficient condition for the existence of a natural De Rham data ; it has been proved in some particular cases (see [19], [32], [9]) and is supported by the existence of a natural connection on periodic cyclic homology (see [13], [42]). The other part, the Betti data, was expected to be given by an hypothetic cohomology theory on non-commutative spaces which is defined over \mathbb{Z} or at least over \mathbb{Q} . It was expected by Toën and Bondal ([22, 2.2.6]) that an appropriate notion of *topological K-theory* for dg-categories over \mathbb{C} can provide a good cohomology theory in order to establish the existence of a \mathbb{Q} -structure on $HP(A)$.

Our work. The purpose of this work is to provide a meaningful definition of *topological K-theory of dg-categories over \mathbb{C}* and to prove that the usual Chern character map from algebraic K-theory to periodic cyclic homology descends naturally to topological K-theory, giving a *topological Chern character map*. The main consequence of these results is the existence of a natural candidate for a \mathbb{Q} -structure on $HP(A)$, given by the image of the topological Chern character.

We now give a quick description of the main definitions and theorems of this work. The definition of topological K-theory is made in two steps : taking (derived) geometric realization of algebraic K-theory, and inverting the Bott element. More precisely if $SPr(Aff/\mathbb{C})$ is the category of simplicial presheaves over affine \mathbb{C} -varieties, and $SSet$ the category of simplicial sets, the geometric realization is a functor

$$|-| : SPr(Aff/\mathbb{C}) \longrightarrow SSet,$$

which extends (via Yoneda) the functor which assigns to a variety the underlying space of its analytification. The geometric realization can be naturally extended to presheaves of spectra. Moreover, there are on both sides of the functor $|-|$ natural model category structures and $|-|$ is left Quillen. The derived functor $\mathbb{L}|-|$ is a key ingredient in our definition of topological K-theory. The presheaves we want to "realize" are given by connective algebraic K-theory denoted by $\tilde{\mathbf{K}}$ and non-connective algebraic K-theory denoted by \mathbf{K} . If T is any \mathbb{C} -dg-category we can now formulate the following definition.

Definition 1.1. *The connective (resp. non-connective) semi-topological K-theory of T is*

$$\tilde{\mathbf{K}}^{\text{st}}(T) := \mathbb{L}|V \mapsto \tilde{\mathbf{K}}(T \otimes_{\mathbb{C}}^{\mathbb{L}} V)| \quad (\text{resp. } \mathbf{K}^{\text{st}}(T) := \mathbb{L}|V \mapsto \mathbf{K}(T \otimes_{\mathbb{C}}^{\mathbb{L}} V)|).$$

The name "semi-topological K-theory" is taken from the work of Friedlander–Walker ([11]) where the authors define such a theory for quasi-projective complex varieties. The first important result we want to emphasize, and which justifies the terminology "topological" is the following.

Theorem 1.2. *(see 4.2) $\tilde{\mathbf{K}}^{\text{st}}(\mathbb{C})$ is naturally isomorphic, in the stable homotopy category of spectra, to the spectrum \mathbf{bu} of connective complex topological K-theory.*

The negative part of the semi-topological K-theory of the point is trivial. This is the content of the second important result of this paper.

Theorem 1.3. *(see 4.3) The natural map $\tilde{\mathbf{K}}^{\text{st}}(\mathbb{C}) \rightarrow \mathbf{K}^{\text{st}}(\mathbb{C})$ is an isomorphism in the stable homotopy category of spectra. Therefore $\mathbf{K}^{\text{st}}(\mathbb{C}) \simeq \mathbf{bu}$. More generally any smooth affine \mathbb{C} -variety has no non-trivial negative semi-topological K-theory.*

These two results enable us to define topological K-theory. We use Tabuada–Cisinski’s Theorem that algebraic K-theory is corepresentable as a functor on Tabuada’s category of "non commutative motives" in order to view $\mathbf{K}^{\text{st}}(T)$ as a module over the ring spectrum $\mathbf{K}^{\text{st}}(\mathbb{C}) \simeq \mathbf{bu}$. If $\beta \in \pi_2 \mathbf{bu}$ is a Bott generator, we define the (non-connective) topological K-theory as

Definition 1.4. $\mathbf{K}^{\text{top}}(T) := \mathbf{K}^{\text{st}}(T)[\beta^{-1}]$.

The philosophy behind this definition goes back to Thomason’s paper [36], where topological K-theory with finite coefficients (for complex algebraic varieties) is recovered from algebraic K-theory with finite coefficients by inverting the Bott element. It is also inspired by Friedlander–Walker’s result on the agreement of Bott inverted semi-topological K-theory with topological K-theory of the geometric realization, for a large class of complex algebraic varieties.

The other important result, with possible applications in non-commutative Hodge theory, is the following.

Theorem 1.5. *(see 4.18) The algebraic Chern character map $\mathbf{K}(T) \rightarrow \mathbf{HC}^-(T)$ descends to topological K-theory providing a homotopy commutative square of spectra*

$$\begin{array}{ccc} \mathbf{K}(T) & \xrightarrow{ch} & \mathbf{HC}^-(T) \\ \downarrow & & \downarrow \\ \mathbf{K}^{\text{top}}(T) & \xrightarrow{ch^{\text{top}}} & \mathbf{HP}(T). \end{array}$$

Another important result of this paper is the agreement of the "classical" definition of $\mathbf{K}^{\text{st}}(T)$ involving the moduli stack \mathcal{M}^T of perfect dg-modules (which is Borel–Moore dual to the moduli stack \mathcal{M}_T of [41]) with our definition of $\mathbf{K}^{\text{st}}(T)$. By definition the space $\mathcal{M}^T(V)$ is the nerve of weak equivalences in the category of perfect dg-modules on $T \otimes_{\mathbb{C}}^{\mathbb{L}} V$. The direct sum of dg-modules endows the object \mathcal{M}^T with a structure of an E_{∞} -monoid in stacks. A remarkable fact we prove in this paper is that the E_{∞} -space $\mathbb{L}|\mathcal{M}^T|$ is group-like.

Proposition 1.6. *(see 4.9) $\tilde{\mathbf{K}}^{\text{st}}(T)$ is weakly equivalent to the spectrum associated to the group-like E_{∞} -space $\mathbb{L}|\mathcal{M}^T|$.*

About proofs. Let us give the main ideas behind the proofs of these results.

The proof of Theorem 1.2 (see subsection 4.2) involves only the definition of algebraic K-theory. Indeed, connective algebraic K-theory can be calculated with vector bundles, and the group completion (with respect to direct sum) of the geometric realization of the stack of vector bundles is equivalent to **bu**.

The proof of Theorem 1.3 is based on the fact that negative K-theory of smooth algebraic varieties vanishes. From this, it suffices to prove that the functor $\mathbb{L}|-|$ satisfies proper hyperdescent, which reduces the proof to the proper hyperdescent theorem in topology. This last fact is the main part of the proof.

The main ideas behind the proof of Theorem 1.5 is to take the geometric realization of the algebraic Chern map, to use the Künneth formula for periodic cyclic homology, and to choose a map from De Rham cohomology to Betti cohomology, which is essentially given by the classical "period map".

The proof of 1.6 is based on proving that a certain map is an \mathbf{A}^1 -equivalence in the sense of Voevodsky, using the nice behaviour of $\mathbb{L}|-|$ with respect to the \mathbf{A}^1 -structure. It is a result of Dugger–Isaksen that the functor $\mathbb{L}|-|$ sends \mathbf{A}^1 -equivalences between cofibrant objects to weak equivalences (see Thm 2.15).

Applications and conjectures. In future works, we hope to formulate a comparison statement about Friedlander–Walker’s semi-topological K-theory of a quasi-projective complex variety and the semi-topological K-theory of the dg-category of perfect complexes on the variety.

Following the analogy with Thomason’s result on Bott inverted algebraic K-theory with finite coefficients, one can formulate a conjecture about the agreement of semi-topological K-theory with finite coefficients with algebraic K-theory with finite coefficients. This is related with the same statement for a certain class of complex algebraic varieties (see [11]), via the comparison with Friedlander–Walker definition.

A possible application of topological K-theory of dg-categories lays in the work of D. Freed [10]. He rises the question whether a Chern–Simons theory can assign to a 1-dimensional closed manifold a linear category, whose dimension reduction would have to lead to a refinement of Hochschild homology defined over the integers. Then he refers to the approach of Bondal and Toën based on topological K-theory, to give a possible explanation for this phenomenon.

Description of the paper. In the first section we recall some basics about Δ and Γ -spaces which are particular models for A_∞ and E_∞ -monoids respectively. Then we recall the definitions and main properties of algebraic K-theory. In the second section we recall the definitions of all variants of cyclic homology and we use Tabuada–Cisinski theory in order to define the Chern character in a linear fashion over the ring spectrum of algebraic K-theory of commutative algebras. Third section is the most original part and is concerned with all that have been summed up in the introduction. In the last part of the paper we state a couple of conjectures about the K-theory of smooth and proper dg-categories which are relevant for non-commutative Hodge theory.

Acknowledgements I am very grateful to Bertrand Toën for accepting me as his student and sharing so generously his mathematical ideas with me. I also want to thank him for proposing me this subject on topological K-theory for my PhD thesis in Montpellier. I want to thank Marco Robalo and Benjamin Hennion for all the comments improving the text.

Notations Because there are no serious set-theoretic problems in this work, we will basically assume that every set is small. This can be made rigorous by choosing universes. If C, D are categories, we denote by C^{op} the opposite category, by $Fun(C, D)$ the category of functors and natural transformations. For any model category M , we denote by $Ho(M)$ its homotopy category. If M is a (cofibrantly generated) model category, and C any category, the category of diagrams $Fun(C, M)$ is by default endowed with the projective model category structure (see [16, 11.6]). We adopt the following notation for the different

mapping objects. If C is just a category, the notation Hom_C refers as usual to the set of morphisms (or maps) in C . If C has moreover an enriched Hom in a category V , we denote it by $\underline{\text{Hom}}_C$ this enriched Hom, and if C is moreover a model category, the derived version of the enriched Hom is denoted by $\mathbb{R}\underline{\text{Hom}}_C$. We omit V from the notation, but each time we need to precise the category V . In the particular case $V = \text{SSet}$, which occurs more frequently than others, we will denote the enriched Hom by Map and its derived version by $\mathbb{R}\text{Map}$. We denote by Δ the standart category of simplices. Let SSet denotes the usual model category of simplicial sets. Let Top denotes the usual model category of topological spaces. Let SGP be the category of group objects in SSet . Let Sp^Σ denotes the stable closed symmetric monoidal model category of symmetric spectra (of simplicial sets) (see [18], [29]). Let WCat denotes the category of Waldhausen categories and exact functors between them. If C is a Waldhausen category we denote by wC its subcategory of weak equivalences.

All along this work, we fix an associative commutative unital base ring k . Let $\text{Aff}/_k$ be the category of affine k -schemes of finite type, $\text{CAlg}/_k$ the category of commutative k -algebras of finite type. We denote by $\text{Pr}(\text{Aff}/_k, C) = \text{Fun}(\text{Aff}/_k^{\text{op}}, C) \simeq \text{Fun}(\text{CAlg}/_k, C)$ the category of presheaves with values in C . We set the notations $\text{SPR}(k) = \text{Pr}(\text{Aff}/_k, \text{SSet})$, the category of simplicial presheaves over $\text{Aff}/_k$. We denote by $\text{Sp}^\Sigma(k) = \text{Pr}(\text{Aff}/_k, \text{Sp}^\Sigma)$ the category of presheaves of symmetric spectra over $\text{Aff}/_k$. By default, the word scheme means scheme of finite type over the base.

Let's denote by $C(k)$ the category of (unbounded) complexes of k -modules. Unless otherwise specified our complexes are written cohomologically, ie are cochain complexes, so that the differential is a map of degree $+1$. Let $\text{dgcCat}/_k$ be the category of k -dg-categories, i.e. of $C(k)$ -enriched categories. By default, if we refer to an object as a dg-category, it means a k -dg-category. For reminders about dg-categories and dg-modules over them, we refer to [24], [40], [39], [41]. If T is a dg-category, we denote by $[T]$ its homotopy category, ie the category with the same objects as T and with hom sets the H^0 of those of T , and with the obvious composition and identities. We denote by $T\text{-Mod}$ the category of left T -dg-modules (ie the category of dg-functors $T \rightarrow C(k)$), endowed with the projective model structure of [39]. It is a stable model category. We denote by $D(T)$ the (triangulated) homotopy category of $T^{\text{op}}\text{-Mod}$. We denote by \widehat{T} the dg-category of *cofibrant* T^{op} -dg-modules. By definition we have an equivalence $[\widehat{T}] \simeq D(T)$. As every representable dg-module is cofibrant, we have a Yoneda embedding

$$\underline{h} : T \rightarrow \widehat{T}.$$

We denote by \widehat{T}_{pe} the dg-category of cofibrant perfect T^{op} -dg-modules, ie of cofibrant dg-modules which are homotopically finitely presented in $T^{\text{op}}\text{-Mod}$ (see [41]). The Yoneda embedding takes value in \widehat{T}_{pe} . We denote by $D_{pe}(T)$ the triangulated subcategory of $D(T)$ which consists of perfect dg-modules, it is the smallest thick triangulated subcategory of $D(T)$ which contains the essential image of the Yoneda embedding $[T] \rightarrow [\widehat{T}]$. We have an equivalence $[\widehat{T}_{pe}] \simeq D_{pe}(T)$.

Model structures on $\text{dgcCat}/_k$ (see [33]) :

- The standard model structure (or Dwyer-Kan model structure), where weak equivalences are quasi-equivalences. We denote by $\text{Ho}(\text{dgcCat}/_k)$ the corresponding homotopy category.
- The Morita model structure, denoted by $\text{dgMor}/_k$, where weak equivalences are the derived Morita equivalences. By definition a dg-functor $f : T \rightarrow T'$ is called a derived Morita equivalence if the induced triangulated functor $\mathbb{L}f_! : D(T) \rightarrow D(T')$ is an equivalence. Or equivalently if the induced triangulated functor $\mathbb{L}f_! : D_{pe}(T) \rightarrow D_{pe}(T')$ is an equivalence. We denote by $\text{Ho}(\text{dgMor}/_k)$ the corresponding homotopy category.

A sequence of dg-categories $T' \xrightarrow{i} T \xrightarrow{p} T''$ in $\text{dgcCat}/_k$ is called *exact* if i is a kernel in $\text{Ho}(\text{dgMor}/_k)$ and p is a cokernel in $\text{Ho}(\text{dgMor}/_k)$, or equivalently if the induced sequence of triangulated categories $D(T') \rightarrow D(T) \rightarrow D(T'')$ is exact (up to factors). A sequence $T' \xrightarrow{i} T \xrightarrow{p} T''$ is called *split exact* if it is

exact and if i has a right adjoint r and p has a right adjoint s such that $r \circ i \simeq id_{T'}$ and $p \circ s \simeq id_{T''}$ via the (co)unit of the adjunctions, as morphisms in $Ho(dgMor/k)$.

2 Preliminaries

2.1 Δ -spaces, Γ -spaces and connective spectra

We will use particular models for A_∞ and E_∞ -monoids known as Δ and Γ -spaces. We recall basic results about Δ -spaces, Γ -spaces, group completion and the link between the homotopy theory of very special Γ -spaces and the homotopy theory connective spectra. Let Γ be the skeletal category of finite pointed sets with objects the sets $n^+ = \{0, \dots, n\}$ with 0 as basepoints for all $n \in \mathbb{N}$ and with morphisms all pointed maps of sets.

Definition 2.1. *Let M be a model category.*

- A Δ -object (resp. a Γ -object) in M is a functor $\Delta^{op} \rightarrow M$ sending $[0]$ to $*$ (resp. a functor $\Gamma \rightarrow M$ sending 0^+ to $*$). Morphisms being natural transformations of functors we denote by $\Delta - M$ (resp. by $\Gamma - M$) the category of Δ -objects (resp. of Γ -objects) in M . For $E \in \Delta - M$ (resp. $F \in \Gamma - M$), we adopt the following notations $E([n]) = E_n$ and $F(n^+) = F_n$.

- A Δ -object E in M is called special if all the Segal maps are weak equivalences in M , ie if for all $[n] \in \Delta$ the map

$$p_0^* \times \dots \times p_{n-1}^* : E_n \longrightarrow E_1^{\times n} = E_1 \overset{h}{\times} \dots \overset{h}{\times} E_1$$

is a weak equivalence in M where $p_i : [1] \rightarrow [n]$, $p_i(0) = i$ and $p_i(1) = i + 1$. We denote by $s\Delta - M$ the full subcategory of $\Delta - M$ consisting of special Δ -objects in M .

- A Γ -object F in M is called special for all $n^+ \in \Gamma$ the map

$$q_*^1 \times \dots \times q_*^n : F_n \longrightarrow F_1^{\times n}$$

is a weak equivalence in M , where $q^i : n^+ \rightarrow 1^+$, $q^i(j) = 1$ if $j = i$ and $q^i(j) = 0$ if $j \neq i$. We denote by $s\Gamma - M$ the full subcategory of $\Gamma - M$ consisting of special Γ -objects in M .

- If $E \in s\Delta - M$, we say that E is very special if the map

$$p_0^* \times d_1^* : E_2 \longrightarrow E_1 \overset{h}{\times} E_1$$

is a weak equivalence in M , where $d_1 : [1] \rightarrow [2]$ is the face map which avoids 1 in $[2]$. We denote by $vs\Delta - M$ the full subcategory of $s\Delta - M$ consisting of very special Δ -objects.

- If $F \in s\Gamma - M$, we say that F is very special if the map

$$q_*^1 \times \mu_* : F_2 \longrightarrow F_1 \overset{h}{\times} F_1$$

is a weak equivalence in M , where $\mu : 2^+ \rightarrow 1^+$ is the map defined by $\mu(1) = 1$ and $\mu(2) = 1$. We denote by $vs\Gamma - M$ the full subcategory of $s\Gamma - M$ consisting of very special Γ -objects.

Remark 2.2. • If we take $M = SSet$, the Δ -objects and Γ -objects are usually called Δ -spaces and Γ -spaces, e.g. in [2].

- The special Δ -objects in M are particular models for "up to coherent homotopy" associative monoids in M (or A_∞ -monoids). If M is replaced by the category of small sets with isomorphisms of sets as weak equivalences, then the category $s\Delta - M$ is equivalent to the category of monoids in sets via the functor evaluation at $[1]$. The composition law of the monoid is recovered by the face map d_1 . And similarly the special Γ -objects are particular models for "up to coherent homotopy" commutative monoids in M (or E_∞ -monoids). The composition law is recovered by the map μ , and the commutativity is encoded by the map $2^+ \rightarrow 2^+$ interchanging 1 and 2.
- For our use of the notion of Δ or Γ -objects, the model category M will in all cases be a quite nice model category in which we can detect weak equivalences by looking what happens on some homotopy groups. For example M will be a category of simplicial presheaves on a category, endowed with the global model structure, or M will be the category of special Γ -objects in simplicial presheaves (where the weak equivalences are taken levelwise). In this situation, the condition of being special and very special can be easily verified. First, we can replace homotopy products by products. And second, the condition of being very special can be reduced at looking what happens on the π_0 as shown in Lemma 2.3. \diamond

Lemma 2.3. *Let $M = SPr(C)$ be the model category of simplicial presheaves on a category C , with the global model structure. If $E \in s\Delta - M$, then E is very special if and only if the monoid $\pi_0 E_1$ is a group. The same works for Γ -objects.*

Proof. Suppose E is very special, then the map

$$p_0^* \times d_1^* : \pi_0 E_2 \longrightarrow \pi_0 E_1 \times \pi_0 E_1$$

is an isomorphism. But if we identify $\pi_0 E_2$ with $\pi_0 E_1 \times \pi_0 E_1$ via the speciality condition then this map sends (a, b) to (a, ab) . This implies that the monoid $\pi_0 E_1$ is a group. Now suppose that $\pi_0 E_1$ is a group. We have to show that the map

$$p_0^* \times d_1^* : E_2 \longrightarrow E_1 \times E_1$$

is a weak equivalence. First this map induces an isomorphism on π_0 because $\pi_0 E_1$ is a group. And second we have to show that it induces an isomorphism on all the π_i for every basepoint and every $i \geq 1$. But all the presheaves of groups $\pi_i(E_1, *)$ are endowed with a monoid law coming from the fact that E is special. Hence these are monoids in the category of presheaves of groups, hence the two laws are the same and the maps $\pi_i(p_0^* \times d_1^*, *)$ are isomorphisms. A similar argument works for Γ -objects. \square

Recall that we have at least three interesting model structures on $\Delta - M$ for any left proper combinatorial model category M :

- The *projective* or *strict model structure* for which weak equivalences and fibrations are levelwise weak equivalences and levelwise fibrations respectively. We denote by $\Delta - M$ this model structure. In all the sequel the expression weak equivalence in $\Delta - M$ will mean levelwise weak equivalences.
- The *special model structure* which is the Bousfield localization of the strict one with respect to the set of maps $(\sqcup_{i=0}^{n-1} h_{p_i} : h_{[1]} \sqcup \dots \sqcup h_{[1]} \longrightarrow h_{[n]})_{n \geq 1}$ \square (generating cofibrations of M). We denote it by $\Delta - M^{sp}$. The fibrant objects in this model structure are the fibrants for the strict model structure which are moreover special.
- The *very special model structure* which is a Bousfield localization of the special one with respect to the map $(h_{p_0} \sqcup h_{d_1} : h_{[1]} \sqcup h_{[1]} \longrightarrow h_{[2]})$ \square (generating cofibrations of M). We denote it by $\Delta - M^{vsp}$. The fibrant objects in this model structure are the fibrants for the strict one which are moreover very special.

We have left derived identity functors

$$Ho(\Delta - M) \xrightarrow{\mathbb{L}id} Ho(\Delta - M^{sp}) \xrightarrow{\mathbb{L}id} Ho(\Delta - M^{vsp})$$

which are denoted by

$$Ho(\Delta - M) \xrightarrow{mon} Ho(\Delta - M^{sp}) \xrightarrow{(-)^+} Ho(\Delta - M^{vsp})$$

and known as "free monoid" and "group completion" respectively.

We have similar model structures for Γ -objects. Indeed, it needs to replace the maps $\sqcup_{i=0}^{n-1} h_{p_i}$ by the maps $\sqcup_{i=1}^n h^{q^i}$ for all $n \geq 1$ and the map $h_{p_0} \sqcup h_{d_1}$ by the map $h^{q^0} \sqcup h^\mu$. We then have a projective model structure $\Gamma - M$, a special model structure $\Gamma - M^{sp}$, and a very special model structure $\Gamma - M^{vsp}$ with corresponding "free abelian monoid" and "abelian group completion" functors :

$$Ho(\Gamma - M) \xrightarrow{com} Ho(\Gamma - M^{sp}) \xrightarrow{(-)^+} Ho(\Gamma - M^{vsp}) .$$

Remark 2.4. Working with $M = SSet$ or with the global model category of simplicial presheaves on a category we have the following. By Segal's Theorem [31, Prop 1.5] the group completion functor $(-)^+$ has as model the composite functor

$$Ho(\Delta - SSet^{sp}) \xrightarrow{|\cdot|} Ho(SSet_*) \xrightarrow{\Omega_\bullet} Ho(\Delta - SSet^{vsp}) ,$$

where $|\cdot|$ is the realization of bisimplicial sets and for a pointed fibrant simplicial set (X, x) the simplicial set $\Omega_n X$ is the simplicial set of maps from Δ^n to X which send all vertices on x . We have indeed more : the composite functor $(-)^+ \circ mon$ has as model the functor $\Omega_\bullet \circ |\cdot|$. \diamond

Example 2.5. The following example will be important to us in this paper on K-theory. If C is any Waldhausen category, we have a Δ -space

$$\mathcal{K}_\bullet(C) := NwS_\bullet C$$

where Nw is the nerve of weak equivalences and S_\bullet is Waldhausen's S-construction. The level 1 is $NwS_1 C$ which is equivalent to NwC . This Δ -space is not special in general. Algebraic K-theory is indeed a way to make it special and moreover very special. The algebraic K-theory space of C is defined by the pointed simplicial set

$$K(C) := \Omega[NwS_\bullet C],$$

where Ω means Ω_1 in the notation of 2.4, ie the simplicial set of loops. The basepoint is taken to be the zero object of C . Hence the K-theory of C is the level 1 of the group completion

$$(mon \mathcal{K}_\bullet(C))^+ \simeq \Omega_\bullet[NwS_\bullet C].$$

Moreover we have

$$\pi_0(mon \mathcal{K}_\bullet(C))_1^+ \simeq (mon \pi_0 \mathcal{K}^{(1)}(C))^+.$$

The free monoid of $\pi_0 \mathcal{K}^{(1)}(C)$ is the monoid in which we identify a with the product of a' and a'' each time there is a cofibration sequence $a' \hookrightarrow a \twoheadrightarrow a''$. It follows that this product is commutative and coincides with the sum in C . Then the group completion of this monoid is the abelian group $K_0(C)$ which is the free abelian group on equivalence classes of objects modulo the relation which identify a with the sum of a' and a'' each time there is a cofibration sequence $a' \hookrightarrow a \twoheadrightarrow a''$. \diamond

We have a fully faithful functor from homotopy commutative monoids to homotopy associative monoids given by the dual of the functor

$$\alpha : \Delta^{op} \longrightarrow \Gamma,$$

defined on objects by $\alpha([n]) = n^+$. And for any map $f : [n] \longrightarrow [m]$ in Δ we define $\alpha(f) = g : m^+ \longrightarrow n^+$ by

$$g(i) = \begin{cases} 0 & \text{if } 0 \leq i \leq f(0) \\ j & \text{if } f(j-1) < i \leq f(j) \\ 0 & \text{if } f(n) < i \end{cases}$$

One can verify that $\alpha(p_i) = q^{i+1}$ for $i = 0, \dots, n-1$, and $\alpha(d_1) = \mu$ so that the fully faithful functor

$$\alpha^* : \Gamma - M \longrightarrow \Delta - M,$$

sends special Γ -objects to special Δ -objects and also very special objects to such. Hence we obtain a diagram

$$\begin{array}{ccccc} Ho(\Gamma - M) & \xrightarrow{com} & Ho(\Gamma - M^{sp}) & \xrightarrow{(-)^+} & Ho(\Gamma - M^{vsp}) \\ \alpha^* \downarrow & & \alpha^* \downarrow & & \alpha^* \downarrow \\ Ho(\Delta - M) & \xrightarrow{mon} & Ho(\Delta - M^{sp}) & \xrightarrow{(-)^+} & Ho(\Delta - M^{vsp}) \end{array}$$

The left square is not commutative anymore but we can actually show that the right square is commutative up to canonical isomorphism.

We recall the equivalence between the homotopy theory of very special Γ -spaces and the homotopy theory of connective spectra. This first appeared in Segal's famous paper [31] and was proved in the language of model categories in [2]. Theorem 5.8 of [2] can be directly generalised from Γ -spaces and spectra to Γ -objects in $M = SPr(C)$ and spectra presheaves on C . Moreover, following [29, example 2.39], we can replaced ordinary spectra by symmetric spectra. We denote by $Sp_c^\Sigma(C)$ the subcategory of connective spectra presheaves. We have a pair of adjoint functor

$$\Gamma - SPr(C) \xrightleftharpoons[\mathcal{A}]{\mathcal{B}} Sp_c^\Sigma(C).$$

Recall that a Γ -space can be extend to a functor from symmetric spectra to symmetric spectra. The functor \mathcal{B} is defined on an object $E \in \Gamma - SPr(C)$ by

$$\mathcal{B}E = E(\mathbb{S}),$$

the value of E on the sphere spectrum, which is a connective spectrum for every Γ -object E . This functor is really identical to Segal's functor from special Γ -spaces to spectra, defined using iterations of realization of simplicial spaces. The functor \mathcal{B} preserves weak equivalences between all Γ -spaces, not just cofibrants. The functor \mathcal{A} is defined on an object $F \in Sp_c^\Sigma(C)$ to be the Γ -object

$$n^+ \mapsto \mathcal{A}(F)_n = \text{Map}(\mathbb{S}^{\times n}, F),$$

where Map is the simplicial mapping space in symmetric spectra. One remark that the level 1 of this Γ -object is

$$\mathcal{A}(F)_1 = \text{Map}(\mathbb{S}, F) \simeq \text{Map}(S^0, F_0) \simeq F_0$$

the 0th term of the spectrum.

We endow the category $Sp^\Sigma(C)$ of symmetric spectra presheaves on C with the projective model structure for which the weak equivalences are the levelwise stable weak equivalences of symmetric spectra and the fibrations are the levelwise fibrations.

Theorem 2.6. *The adjoint pair $(\mathcal{B}, \mathcal{A})$ is a Quillen pair for the very special model structure on $\Gamma - \text{SPR}(C)$. Moreover it is a Quillen equivalence, inducing an equivalence of categories*

$$Ho(\Gamma - \text{SPR}(C)^{\text{vsp}}) \xrightleftharpoons[\mathbb{R}\mathcal{A}]{\mathbb{L}\mathcal{B}} Ho(\text{Sp}_c^\Sigma(C)) .$$

Remark 2.7. In fact the functor \mathcal{B} preserves all weak equivalences and need not be derived. \diamond

2.2 Connective algebraic K-theory

We recall how to define the connective algebraic K-theory of a dg-category using Waldhausen's S -construction ([43]) and following [24].

If C is a Waldhausen category, we denote by $K(C)$ its algebraic K-theory space. It is given by the pointed simplicial set

$$K(C) := \Omega |NwS_\bullet C| ,$$

where $|-|$ is the realization of bisimplicial sets and Ω is the loop simplicial set based at 0. It defines a functor

$$K : W\text{Cat} \longrightarrow S\text{Set}_* .$$

In the following we will not use the same delooping as Waldhausen for the K-theory space, instead we will use a more canonical model for the weak homotopy type of the connective algebraic K-theory symmetric spectrum. The idea is that this delooping is given by the sum in the Waldhausen category C instead of given by the iteration of the S -construction. The sum in C is encoded by a special Γ -object in $W\text{Cat}$ denoted by $B_\bullet C$. The level 1 of $B_\bullet C$ is equivalent to C and the level n is equivalent to C^n . Now one can define a special Γ -space by taking the K-theory space levelwise :

$$K^\Gamma(C) := K(B_\bullet C) .$$

Because $\pi_0 K^\Gamma(C)_1 \simeq K_0(C)$ is a group, the Γ -object $K^\Gamma(C)$ is very special and thus gives a connective symmetric spectrum

$$\tilde{K}(C) := \mathcal{B}K^\Gamma(C) .$$

This defines a functor

$$\tilde{K} : W\text{Cat} \longrightarrow \text{Sp}^\Sigma .$$

Let T be any dg-category. We consider $\text{Perf}(T)$ the category of cofibrant and perfect T^{op} -dg-modules ie the sub-category of $T^{\text{op}} - \text{Mod}$ consisting of cofibrant T^{op} -dg-modules which are perfect (or compact) as objects of the derived category $D(T)$. We endow $\text{Perf}(T)$ with a structure of a Waldhausen category (category with cofibrations and weak equivalences in [43]) induced by the model structure of $T^{\text{op}} - \text{Mod}$, i.e. a map is a weak equivalence (resp. a cofibration) in $\text{Perf}(T)$ if it is so in $T^{\text{op}} - \text{Mod}$. The axioms of a Waldhausen category structure are satisfied essentially because the homotopy pushout of two perfect dg-modules over a third perfect dg-module is again perfect. Moreover, the Waldhausen category $\text{Perf}(T)$ satisfies the saturation axiom, the extension axiom, has a cylinder functor which satisfies the cylinder axiom.

Let $f : T \longrightarrow T'$ be a map in dgCat/k , then f induces a Quillen pair

$$T^{\text{op}} - \text{Mod} \xrightleftharpoons[f^*]{f_!} T'^{\text{op}} - \text{Mod}$$

where f^* is defined on objects by composition with f . As a left Quillen functor, the direct image $f_!$ preserves perfect dg-modules and induces an exact functor still denoted by $f_!$

$$f_! : \text{Perf}(T) \longrightarrow \text{Perf}(T') .$$

This defines a lax functor $dgCat_{/k} \longrightarrow WCat$ to which we apply the canonical strictification procedure to obtain a functor

$$Perf : dgCat_{/k} \longrightarrow WCat.$$

Definition 2.8. a) *The algebraic K-theory space functor of dg-categories is the composite functor*

$$K \circ Perf : dgCat_{/k} \longrightarrow WCat \longrightarrow SSet_*,$$

still denoted by K .

b) *The connective algebraic K-theory functor of dg-categories is the composite functor*

$$\tilde{K} \circ Perf : dgCat_{/k} \longrightarrow WCat \longrightarrow Sp^\Sigma,$$

still denoted by \tilde{K} .

Remark 2.9. If $T = A$ is any associative k -algebra, one can consider vector bundles on $Spec(A)$, or in other words, projective (right) A -modules of finite type. This forms a Waldhausen category $Vect(A)$ with weak equivalences being isomorphisms and cofibrations being monomorphisms. One can show (using [35, 1.11.7]) that there is a weak equivalence of simplicial sets $K(Vect(A)) \simeq K(Perf(A))$, and thus a weak equivalence on the associated connective K-theory spectra too.

We now recall the main properties of connective K-theory : filtered colimits, derived Morita invariance and additivity on split short exact sequences of dg-categories.

Proposition 2.10. a) *The functor \tilde{K} commutes with filtered colimits and filtered homotopy colimits in $dgCat_{/k}$.*

b) *The functor \tilde{K} sends derived Morita equivalences in $dgCat_{/k}$ to isomorphisms in $Ho(Sp^\Sigma)$.*

c) *Let $T' \xrightarrow{i} T \xrightarrow{p} T''$ be a **split** short exact sequence of dg-categories. Then the morphism*

$$i_! + p^* : \tilde{K}(T') \oplus \tilde{K}(T'') \longrightarrow \tilde{K}(T)$$

is an isomorphism in $Ho(Sp^\Sigma)$.

Proof. a) The fact that it commutes with colimits can be deduced from Waldhausen approximation theorem, details will be given in [1]. For homotopy colimits, this follows from the fact that $dgCat_{/k}$ is a compactly generated model category in the sense of [41, Def 2.1] and thus by [41, Prop 2.2], for any filtered diagram $(T_i)_{i \in I}$ of dg-categories, the map $hocolim_{i \in I} T_i \longrightarrow colim_{i \in I} T_i$ is an isomorphism in $Ho(dgCat_{/k})$, and we apply b).

b) It is a consequence of Thomason result [35, Thm 1.9.8].

c) Can be deduced from Waldhausen's additivity theorem, details will be given in [1].

□

2.3 Non-connective algebraic K-theory

The preceding K-theory spectrum \tilde{K} is a connective spectrum, ie its negative homotopy groups are trivial. However, a dg-category or even a (singular) scheme can have negative K-theory. Using Schlichting's construction (see[28]), which recovers that of Thomason, we can define this negative K-theory. Schlichting's construction applies to objects called Frobenius pairs (due to B. Keller [23]). Here we use Tabuada-Cisinski's construction of non-connective K-theory [4], which is defined directly on the level

of dg-categories. The relation to Schlichting's construction is made precise by the comparison result [4, Prop 6.6].

The main ingredient of non-connective K-theory is the "countable sum completion functor" or "flasque envelope" :

$$\mathcal{F} : dgCat_{/k} \longrightarrow dgCat_{/k},$$

(see [4] for this construction). It comes with a quasi-fully faithful functor $T \longrightarrow \mathcal{F}(T)$. The essential property of the dg-category $\mathcal{F}(T)$ is to have countable sums, and thus has vanishing \mathbf{K}_0 .

Then one can define the suspension functor of dg-categories

$$\mathcal{S} : dgCat_{/k} \longrightarrow dgCat_{/k},$$

by $\mathcal{S}(T) := \mathcal{F}(T)/T$, the quotient dg-category in $Ho(dgCat_{/k})$. The sequence of spectra $(\tilde{\mathbf{K}}(\mathcal{S}^n(T)))_{n \geq 0}$ forms a spectrum in Sp^Σ (see [4, Prop 7.2]) and we take the 0th-level of the associated Ω -spectrum to define K-theory.

Definition 2.11. ([4, Prop 7.5]) *The (non-connective) algebraic K-theory functor of dg-categories*

$$\mathbf{K} : dgCat_{/k} \longrightarrow Sp^\Sigma,$$

is the functor defined by

$$\mathbf{K}(T) := \text{hocolim}_{n \geq 0} \tilde{\mathbf{K}}(\mathcal{S}^n(T))[-n].$$

By definition, for every $T \in dgCat_{/k}$ we have a natural map $\tilde{\mathbf{K}}(T) \longrightarrow \mathbf{K}(T)$. The main properties of non-connective K-theory are the following. In fact, it can be shown that there is essentially a unique way to extend $\tilde{\mathbf{K}}$ to a non-connective invariant satisfying all these properties.

Proposition 2.12. a) *For every $T \in dgCat_{/k}$, the natural map $\tilde{\mathbf{K}}(T) \longrightarrow \mathbf{K}(T)$ induces an isomorphism on π_i for all $i \geq 0$. Therefore $\tilde{\mathbf{K}}(T)$ is the connective covering of the spectrum $\mathbf{K}(T)$.*

b) *The functor \mathbf{K} commutes with filtered colimits and filtered homotopy colimits in $dgCat_{/k}$.*

c) *The functor \mathbf{K} sends derived Morita equivalences to isomorphism in $Ho(Sp^\Sigma)$.*

d) *Let $T' \xrightarrow{i} T \xrightarrow{p} T''$ be an exact sequence of dg-categories. Then the induced sequence*

$$\mathbf{K}(T') \xrightarrow{i_!} \mathbf{K}(T) \xrightarrow{p_!} \mathbf{K}(T'')$$

is a distinguished triangle in $Ho(Sp^\Sigma)$.

Proof. **a)** comes from the definition and from the fact that $\mathbf{K}(T)$ is an Ω -spectrum.

b) can be deduced from the corresponding assertion for $\tilde{\mathbf{K}}$, see [1] for details.

c) follows from [4, Prop 6.6] and from the corresponding assertion in the context of Frobenius pairs [28, 12.3, Prop 3].

d) as above one reduces to the case of Frobenius pairs by [4, Prop 6.6] and uses [28, 12.1, Thm 9]. □

2.4 Geometric realization over complex numbers

There is a geometric realization functor

$$Aff_{/\mathbb{C}} \xrightarrow{|\cdot|^{top}} Top$$

which associates to every affine schemes of finite type over \mathbb{C} the underlying space of its analytification. We denote by

$$SSet \xrightleftharpoons[S]{Re} Top$$

the standard adjunction with right adjoint the singular functor S . We can compose this geometric realization functor with S to obtain a geometric realization with target the category of simplicial sets

$$Aff_{/\mathbb{C}} \xrightarrow{|\cdot|^{top}} Top \xrightarrow{S} SSet$$

$\searrow |\cdot|$

We denote by $Pr(\mathbb{C})$ the category of presheaves over $Aff_{/\mathbb{C}}$, by $SPr(\mathbb{C})$ the category of simplicial presheaves over $Aff_{/\mathbb{C}}$ and by $Sp^{\Sigma}(\mathbb{C})$ the category of symmetric spectra presheaves over $Aff_{/\mathbb{C}}$. For any kind of presheaves E over $Aff_{/\mathbb{C}}$ we write $E(A)$ for $E(Spec(A))$, where $A \in CAlg_{/\mathbb{C}}$. We have functors

$$Aff_{/\mathbb{C}} \xrightarrow{h} Pr(\mathbb{C}) \xrightarrow{cst} SPr(\mathbb{C}) \xrightarrow{\Sigma^{\infty}(-)_{+}} Sp^{\Sigma}(\mathbb{C})$$

where h is the Yoneda embedding, cst is just to see a set as a constant simplicial set and $\Sigma^{\infty}(-)_{+}$ is the composition of a simplicial presheaf with adding a disjoint basepoint and taking the infinite suspension symmetric spectrum (see [29]). By composition we then have a strictly commutative triangle of two Yoneda embeddings

$$\begin{array}{ccc} Aff_{/\mathbb{C}} & \longrightarrow & SPr(\mathbb{C}) \\ & \searrow & \downarrow \Sigma^{\infty}(-)_{+} \\ & & Sp^{\Sigma}(\mathbb{C}) \end{array}$$

In the sequel, the category $SPr(\mathbb{C})$ is endowed with the so called *global model structure*, ie the projective model structure when seen as a diagram category $SPr(\mathbb{C}) = SSet^{Aff_{/\mathbb{C}}^{op}}$. So by default, the notation $SPr(\mathbb{C})$ refers to this global model structure.

Definition 2.13. The $SSet$ -enriched left Kan extension of the functor $|\cdot|^{top}$ along the Yoneda embedding $Aff_{/\mathbb{C}} \longrightarrow SPr(\mathbb{C})$ is called the geometric realization and still denoted by

$$SPr(\mathbb{C}) \xrightarrow{|\cdot|^{top}} Top .$$

The composition of this new $|\cdot|^{top}$ with S is then the $SSet$ -enriched left Kan extension of $|\cdot| : Aff_{/\mathbb{C}} \longrightarrow SSet$ along the Yoneda embedding, it is denoted by

$$SPr(\mathbb{C}) \xrightarrow{|\cdot|} SSet ,$$

and refer to as the simplicial geometric realization.

We summarize the properties we will need about this geometric realization in a proposition.

Proposition 2.14. *1. The simplicial geometric realization commutes with colimits and has a simplicial right adjoint R which sends a simplicial set K to the simplicial presheaf*

$$R(K) : \text{Spec}(A) \mapsto \text{Map}(|\text{Spec}(A)|, K),$$

where Map is the simplicial mapping space in SSet . This adjunction is a Quillen pair for the global model structure on $\text{SPr}(\mathbb{C})$.

2. The geometric realization commutes with finite limits of simplicial presheaves.

3. For every simplicial set K and every simplicial presheaf E one has a canonical isomorphism in SSet ,

$$K \times |E| \xrightarrow{\sim} |K \times E|.$$

4. For every pointed simplicial set K and every pointed simplicial presheaf E one has a canonical isomorphism in SSet ,

$$K \wedge |E| \xrightarrow{\sim} |K \wedge E|.$$

Proof. 1. We can guess the right adjoint R writing the formula

$$\text{Map}_{\text{SSet}}(|E|, K) \simeq \text{Map}_{\text{SPr}(\mathbb{C})}(E, R(K))$$

and taking E representable by an $A \in \text{CAlg}/\mathbb{C}$, we obtain $R(K)(A) = \text{Map}_{\text{SSet}}(|A|, K)$. The fact that it forms a Quillen pair follows from general nonsense of [6, Prop 1.1].

2. This can be checked on representable presheaves, for which it is straightforward by definition of the geometric realization (using eg that the analytification functor commutes with finite limits [15, exposé XII]).

3. The argument to see 4. is similar. If $E \in \text{SPr}(\mathbb{C})$ and $K, K' \in \text{SSet}$,

$$\begin{aligned} \text{Hom}_{\text{SSet}}(|K \times E|, K') &\simeq \text{Hom}_{\text{SPr}(\mathbb{C})}(K \times E, R(K')) \\ &\simeq \text{Hom}_{\text{SPr}(\mathbb{C})}(E, \text{Map}(K, R(K'))) \\ &\simeq \text{Hom}_{\text{SPr}(\mathbb{C})}(E, \text{Map}(K \times | - |, K')) \\ &\simeq \text{Hom}_{\text{SSet}}(K \times |E|, K'). \end{aligned}$$

□

We now endow the category Aff/\mathbb{C} with a topology, say the étale one to fix ideas. Everything we will say below is valid for the Nisnevich topology as well. Recall (following [27]) that we can build the \mathbf{A}^1 -motivic étale homotopy theory of schemes from the étale site of affine schemes Aff/\mathbb{C} . Let \mathcal{Q} denotes the set consisting of all maps in $\text{SPr}(\mathbb{C})$ which are of the form

1. $\text{hocolim}_{\Delta^{\text{op}}} U_{\bullet} \rightarrow X$, for $U_{\bullet} \rightarrow X$ an étale hypercovering of a scheme $X \in \text{Aff}/\mathbb{C}$, and the homotopy colimit is calculated in $\text{SPr}(\mathbb{C})$.
2. or a projection $E \times \mathbf{A}^1 \rightarrow E$, for $E \in \text{SPr}(\mathbb{C})$.

A model for the \mathbf{A}^1 -étale homotopy theory of schemes over \mathbb{C} is given by the left Bousfield localization $L_{\mathcal{Q}}\text{SPr}(\mathbb{C}) =: \text{SPr}(\mathbb{C})^{\text{ét}, \mathbf{A}^1}$ of the global model category of simplicial presheaves over \mathbb{C} by the set \mathcal{Q} . Weak equivalences in $\text{SPr}(\mathbb{C})^{\text{ét}, \mathbf{A}^1}$ are called \mathbf{A}^1 -equivalences. One of the most important property of the geometric realization is the following fact taken from [8].

Proposition 2.15 (Dugger-Isaksen [8] Thm 5.2). *The geometric realization is the left Quillen functor of a Quillen pair*

$$SPr(\mathbb{C})^{\text{ét}, \mathbf{A}^1} \xrightleftharpoons{|-|^{top}} Top .$$

Proof. By general non sense about Bousfield localizations [16, Thm 3.3.20] it suffices to show that the geometric realization sends relations 1 and 2 defining \mathcal{Q} to weak equivalences of topological spaces. Let $U_\bullet \rightarrow X$ be an étale hypercovering of an affine scheme X . The map $|U_\bullet|^{top} \rightarrow |X|^{top}$ is a generalized topological hypercovering, hence by [8, Prop 4.10], the map $hocolim_{\Delta^{op}} |U_\bullet|^{top} \rightarrow |X|^{top}$ is a weak equivalence of spaces. Because the canonical map

$$hocolim_{\Delta^{op}} |U_\bullet|^{top} \rightarrow |hocolim_{\Delta^{op}} U_\bullet|^{top}$$

is a weak equivalence of spaces, this show that the map $|hocolim_{\Delta^{op}} U_\bullet|^{top} \rightarrow |X|^{top}$ is a weak equivalence. Next, the maps $|E \times \mathbf{A}^1|^{top} \rightarrow |E|^{top}$ are sent to weak equivalences because $|-|^{top}$ commutes with products and because $|\mathbf{A}^1|^{top} = \mathbb{C}$ is contractible. \square

Remark 2.16. • It follows from 2.15 that the simplicial geometric realization $|-|$ is also a left Quillen functor.

- Again it follows from 2.15 that the derived geometric realization $\mathbb{L}|-|^{top}$ preserves weak equivalences, hence sends \mathbf{A}^1 -equivalences to weak equivalences. The non derived geometric realization fails to preserves all weak equivalences just because of the cofibrancy condition in the global model structure on simplicial presheaves $SPr(\mathbb{C})$. It is because the functor $|-|^{top}$ preserves relations 1. and 2. that define the model category $SPr(\mathbb{C})^{\text{ét}, \mathbf{A}^1}$. Therefore for any simplicial presheaf $E \in SPr(\mathbb{C})^{\text{ét}, \mathbf{A}^1}$, one has $\mathbb{L}|E|^{top} = |QE|^{top}$, where Q is a cofibrant replacement functor in the global model category $SPr(\mathbb{C})$. Consequently, in the sequel we don't precise the source homotopy category of "the" derived geometric realization $\mathbb{L}|-|^{top}$. \diamond

The proof of the following is similar to the non-derived version, so we omit it.

Proposition 2.17. *The derived geometric realization $\mathbb{L}|-|$ enjoys the property 1,3,4 of 2.14, where we replace product by derived product and smash by derived smash. The right adjoint is denoted by $(-)_B$ (like "Betti"). It is given by $K_B = \mathbb{R}Map_{Ho(SSet)}(|-|, K)$. Moreover the functor $\mathbb{L}|-|$ commutes with finite homotopy products in $Ho(SPr(\mathbb{C}))$.*

Let $SGp(\mathbb{C})$ denotes the category of group objects in the category $SPr(\mathbb{C})$. We have the notion of levelwise weak equivalence in $SGp(\mathbb{C})$ as in $SPr(\mathbb{C})$. One has the classifying space functor

$$B : Ho(SGp) \longrightarrow Ho(SSet)$$

defined in the following way. Every simplicial group G gives rise to a bisimplicial set $G^\bullet = ([n] \mapsto G^n)$ and BG is defined by

$$BG := hocolim G^\bullet .$$

For $G \in SGp(\mathbb{C})$, one sets $(BG)(A) := B(G(A))$. Because $\mathbb{L}|-|$ commutes in fact with products, the geometric realization of a group simplicial presheaf is a simplicial group. We then have the diagram

$$\begin{array}{ccc} Ho(SGp(\mathbb{C})) & \xrightarrow{B} & Ho(SPr(\mathbb{C})) \\ \downarrow \mathbb{L}|-| & & \downarrow \mathbb{L}|-| \\ Ho(SGp) & \xrightarrow{B} & Ho(SSet) \end{array}$$

Proposition 2.18. *For every $G \in SGP(\mathbb{C})$, one has a canonical isomorphism $B\mathbb{L}|G| \simeq \mathbb{L}|BG|$ in $Ho(SSet)$.*

Proof. It comes from the fact that $\mathbb{L}| - |$ commutes with homotopy finite products and that it commutes with homotopy colimits. \square

We now give a convenient description of the set $\pi_0\mathbb{L}|E|$ for any simplicial presheaf $E \in SPr(\mathbb{C})$. It will be useful below. It is based on the following formula. Let $F \in Pr(\mathbb{C})$ be any presheaf of sets on Aff/\mathbb{C} , then we have a weak equivalence in $SPr(\mathbb{C})$

$$hocolim_{X \in Aff/F} X \xrightarrow{\sim} F ,$$

where $Aff/\mathbb{C}/F$ is the category of complex affine schemes over F .

Proposition 2.19. *Let $F \in Pr(\mathbb{C})$ be a presheaf of sets. There exists a canonical isomorphism of sets*

$$F(\mathbb{C}) / \sim \xrightarrow{\sim} \pi_0\mathbb{L}|F| ,$$

where the equivalence relation \sim is defined by $[x] \sim [y]$ if there exists a connected algebraic curve C , a morphism $f : C \rightarrow F$ in $Ho(SPr(\mathbb{C}))$, and two complex points $x', y' \in C(\mathbb{C})$ such that $f(x') = x$ and $f(y') = y$.

Proof. The weak equivalence $F \simeq hocolim_{X \in Aff/F} X$ gives isomorphisms of sets $F(\mathbb{C}) \simeq hocolim_{X \in Aff/F} X(\mathbb{C}) \simeq colim_{X \in Aff/F} X(\mathbb{C})$ and $\pi_0\mathbb{L}|F| \simeq colim_{X \in Aff/F} \pi_0|X|$. We then have a commutative square

$$\begin{array}{ccc} F(\mathbb{C}) & \xrightarrow{\quad} & \pi_0\mathbb{L}|F| \\ \downarrow \wr & & \downarrow \wr \\ colim_{X \in Aff/F} X(\mathbb{C}) & \longrightarrow & colim_{X \in Aff/F} \pi_0|X| \end{array}$$

where the bottom arrow is induced by the maps $X(\mathbb{C}) \rightarrow \pi_0|X|$, which are all surjective. Therefore the top arrow is surjective. Suppose now that two elements $a, b \in F(\mathbb{C})$ are identified by the top arrow. Then a and b corresponds to pairs (X, x) and (Y, y) with X, Y affine schemes over F , $x \in X(\mathbb{C})$ and $y \in Y(\mathbb{C})$ complex points mapping to a and b respectively. Thus (X, x) and (Y, y) are identified in $colim_{X \in Aff/F} \pi_0|X|$. Therefore there exists an affine scheme $Z \in Aff/\mathbb{C}$ over F , morphisms $p : X \rightarrow Z$ and $q : Y \rightarrow Z$ and a continuous path joining $p(x)$ to $q(y)$ in $|Z|^{top}$. It's a well known fact in algebraic geometry that in this situation, using Bertini's theorem one can show that there exists a morphism $g : C \rightarrow Z$, where C is a connected algebraic curve (with at most nodal points), and complex points $x', y' \in C(\mathbb{C})$ such that $g(x') = p(x)$ and $g(y') = q(y)$. By composition we then have a morphism $f : C \rightarrow F$ such that $f(x') = a$ and $f(y') = b$. This observation finishes the proof of 2.19. \square

Corollary 2.20. *Let $E \in SPr(\mathbb{C})$. There exists a canonical isomorphism of sets*

$$\pi_0 E(\mathbb{C}) / \sim \xrightarrow{\sim} \pi_0\mathbb{L}|E| ,$$

where the equivalence relation \sim is defined by $[x] \sim [y]$ if there exists a connected algebraic curve C , a morphism $f : C \rightarrow E$ in $Ho(SPr(\mathbb{C}))$, and two complex points $x', y' \in C(\mathbb{C})$ such that $f(x') = x$ and $f(y') = y$ in $\pi_0 E(\mathbb{C})$.

Remark 2.21. Recall that the functor π_0 for simplicial sets is by definition the π_0 of a fibrant replacement.

Proof. We apply Proposition 2.19 to the presheaf of sets $F := \pi_0^{pr} E$. Then we get the desired isomorphism. \square

Remark 2.22. 1. The Sp^Σ -enriched left Kan extension of $|-| : SPr(\mathbb{C}) \longrightarrow SSet$ along $SPr(\mathbb{C}) \longrightarrow Sp^\Sigma(\mathbb{C})$ exists and is denoted by

$$|-| : Sp^\Sigma(\mathbb{C}) \longrightarrow Sp^\Sigma.$$

2. Its right adjoint is the functor which sends $F \in Sp^\Sigma$ to $\underline{\text{Hom}}_{Sp^\Sigma}(\Sigma^\infty|-|_+, F) \in Sp^\Sigma(\mathbb{C})$. The right adjoint to the derived functor $\mathbb{L}|-|$ is denoted by $F \longmapsto F_B$.

3. For every $F \in Sp^\Sigma$ and every $E \in Sp^\Sigma(\mathbb{C})$ we have a canonical isomorphism in Sp^Σ

$$|F \wedge E| \simeq F \wedge |E|.$$

In an other direction, the geometric realization can be extended to Δ -objects (resp. Γ -objects) in $SPr(\mathbb{C})$ with values in Δ -spaces (resp. in Γ -spaces). We write it for Δ -objects and the same works for Γ -objects. This is done by taking simplicial realization levelwise, ie if $E \in \Delta - SPr(\mathbb{C})$, $|E|$ is the Δ -space $[n] \mapsto |E_n|$. This formula defines a functor still denoted by

$$\Delta - SPr(\mathbb{C}) \xrightarrow{|-|} \Delta - SSet$$

whose right adjoint sends a Δ -space F to the Δ -presheaf

$$(A, [n]) \longmapsto \text{Map}(|A|, F_n).$$

By 2.14, the derived geometric realization commutes with homotopy finite limits, and thus sends special Δ -objects to special Δ -objects and it sends very special Δ -objects to very special Δ -objects. Hence we have a diagram of categories

$$\begin{array}{ccc} Ho(\Delta - SPr(\mathbb{C})) & \xleftarrow{\mathbb{L}|-|} & Ho(\Delta - SSet) \\ \uparrow \downarrow \text{mon} & & \uparrow \downarrow \text{mon} \\ Ho(\Delta - SPr(\mathbb{C})^{sp}) & \xleftarrow{\mathbb{L}|-|} & Ho(\Delta - SSet^{sp}) \\ \uparrow \downarrow (-)^+ & & \uparrow \downarrow (-)^+ \\ Ho(\Delta - SPr(\mathbb{C})^{vsp}) & \xleftarrow{\mathbb{L}|-|} & Ho(\Delta - SSet^{vsp}) \end{array}$$

Proposition 2.23. For every $E \in \Delta - SPr(\mathbb{C})$, one has a canonical isomorphism

$$\mathbb{L}|mon E| \simeq mon \mathbb{L}|E|$$

in $Ho(\Delta - SSet)$.

Proof. It is equivalent to prove the commutativity property for the right adjoints to these functors, for which it is true because for every $F \in s\Delta - SSet$, the Δ -object $[n] \mapsto \mathbb{R}\text{Map}(|-|, F_n)$ is special, where the $\mathbb{R}\text{Map}$ is taken in the projective model structure $\Delta - SSet$. \square

Proposition 2.24. For every $E \in \Delta - SPr(\mathbb{C})$, one has a canonical isomorphism

$$\mathbb{L}|E^+| \simeq (\mathbb{L}|E|)^+,$$

in $Ho(\Delta - SSet)$.

Proof. It is equivalent to prove the commutativity property for the right adjoints to these functors, for which it is true because for $F \in vs\Delta - SSet$, the Δ -object $[n] \mapsto \mathbb{R}\text{Map}(|-|, F_n)$ is very special. \square

On the level of Γ -objects, one has similar diagrams with respect to special and very special Γ -objects, and the analog of Proposition 2.24. The right adjoint to the realization

$$\Gamma - SPr(\mathbb{C}) \xrightarrow{|\cdot|} \Gamma - SSet$$

sends a Γ -space F to the Γ -presheaf $\underline{\text{Hom}}_{\Gamma - SSet}(|\cdot|, F)$, which is the internal Hom of $\Gamma - SSet$ defined by $\underline{\text{Hom}}_{\Gamma - SSet}(E, F)_n = \text{Map}(E, F(n^+ \wedge -))$, where $\wedge : \Gamma \times \Gamma \rightarrow \Gamma$ is the monoidal structure on Γ given by the wedge of pointed sets. The object $\underline{\text{Hom}}_{\Gamma - SSet}(|\cdot|, F)$ is isomorphic to the Γ -presheaf

$$(A, n^+) \mapsto \text{Map}(|A|, F_n).$$

Combining this realization of Γ -presheaves with the adjoint pair $(\mathcal{B}, \mathcal{A})$ we obtain a diagram

$$\begin{array}{ccc} \Gamma - SPr(\mathbb{C}) & \xrightleftharpoons[\mathcal{A}]{\mathcal{B}} & Sp_c^\Sigma(\mathbb{C}) \\ \uparrow \downarrow |\cdot| & & \uparrow \downarrow |\cdot| \\ \Gamma - SSet & \xrightleftharpoons[\mathcal{A}]{\mathcal{B}} & Sp_c^\Sigma \end{array}$$

Proposition 2.25. *For every $E \in \Gamma - SPr(\mathbb{C})$ one has a canonical isomorphism*

$$\mathcal{B}|E| \simeq |\mathcal{B}E|$$

in Sp_c^Σ , hence a canonical isomorphism

$$\mathcal{B}\mathbb{L}|E| \simeq \mathbb{L}|\mathcal{B}E|$$

in $Ho(Sp^\Sigma)$.

Proof. It is equivalent to prove the commutativity property for the right adjoint to these two functors, for which is says that for $X \in Sp_c^\Sigma$ one has a canonical isomorphism

$$\mathcal{A}\underline{\text{Hom}}_{Sp^\Sigma}(|\cdot|, X) \simeq \underline{\text{Hom}}_{\Gamma - SSet}(|\cdot|, \mathcal{A}X).$$

One verify the existence of such an isomorphism using the explicit formula for \mathcal{A} and the following sequence of simplicial adjunctions

$$\begin{aligned} \mathcal{A}\underline{\text{Hom}}_{Sp^\Sigma}(|\cdot|, X)_n &\simeq \text{Map}(\mathbb{S}^{\times n}, \underline{\text{Hom}}_{Sp^\Sigma}(|\cdot|, X)) \\ &\simeq \text{Map}(\mathbb{S}^{\times n} \wedge \Sigma^\infty | - |_+, X) \\ &\simeq \text{Map}(\Sigma^\infty | - |_+, \underline{\text{Hom}}_{Sp^\Sigma}(\mathbb{S}^{\times n}, X)) \\ &\simeq \text{Map}(|\cdot|, \text{Map}_{Sp^\Sigma}(\mathbb{S}^{\times n}, X)) \\ &\simeq \text{Map}(|\cdot|, (\mathcal{A}X)_n) \\ &\simeq \underline{\text{Hom}}_{\Gamma - SSet}(|\cdot|, \mathcal{A}X)_n. \end{aligned}$$

□

Proposition 2.26. *For every $E \in \Gamma - SPr(\mathbb{C})$, one has a canonical isomorphism*

$$|\alpha^* E| \simeq \alpha^* |E|$$

in $\Delta - SSet$, hence a canonical isomorphism

$$\mathbb{L}|\alpha^* E| \simeq \alpha^* \mathbb{L}|E|$$

in $Ho(\Delta - SSet)$.

Proof. It comes from the definition of the functor α^* . □

To end this section, let's state some homotopical properties of the realization of Δ and Γ -presheaves over \mathbb{C} . It is a direct consequence of Proposition 2.15. We endow the category $\Delta - \mathcal{S}Pr(\mathbb{C})^{\text{ét}, \mathbf{A}^1}$ (resp. $\Gamma - \mathcal{S}Pr(\mathbb{C})^{\text{ét}, \mathbf{A}^1}$) with the projective model structure in the Δ (resp. Γ) -direction.

Proposition 2.27. *The adjoint pairs*

$$\Delta - \mathcal{S}Pr(\mathbb{C})^{\text{ét}, \mathbf{A}^1} \xrightleftharpoons{|-|} \Delta - \mathcal{S}Set, \quad \Gamma - \mathcal{S}Pr(\mathbb{C})^{\text{ét}, \mathbf{A}^1} \xrightleftharpoons{|-|} \Gamma - \mathcal{S}Set$$

are Quillen pairs for the projective model structure on categories of Δ and Γ -objects.

3 Algebraic Chern character

3.1 Cyclic homologies spectra

The different versions of cyclic homology are all defined starting with the mixed complex associated to a dg-category, as defined by Keller [23]. This is a functor

$$Mix = Mix(- \mid k) : dgCat_{/k} \longrightarrow \Lambda - Mod.$$

where the target is the category of dg-modules over the k -dg-algebra Λ generated by one element B in degree -1 , submitted to the relation $B^2 = 0$ and $d(B) = 0$. A Λ -dg-module is also called a mixed complex. It is a relative invariant in the sense that it depends of the base. However for convenience, we will forget to mention the base ring and in all the sequel, otherwise specified, the mixed complex and cyclic homologies are calculated over the base ring k . Keller proved that Mix is a localizing invariant.

Theorem 3.1 (Keller [23]). *The functor Mix commutes with filtered (homotopy) colimits (in $dgCat_{/k}$), sends derived Morita equivalences to quasi-isomorphisms and short exact sequences of dg-categories to distinguished triangles in the derived category of mixed complexes.*

We denote by

$$H : C(\mathbb{Z}) \longrightarrow Sp^{\Sigma}$$

the standard functor, see [1] for precisions. The categories $C(\mathbb{Z})$ and Sp^{Σ} are monoidal categories, for usual graded tensor product of complexes and smash product respectively. The functor H is a lax monoidal functor and so it induces functors between monoids and modules over that monoids. Hence we have a ring spectrum Hk , an Hk -algebra spectrum $H\Lambda$, and the functor H induces a functor between categories of modules still denoted by

$$H : \Lambda - Mod \longrightarrow H\Lambda - Mod^{\Sigma}.$$

We endow $\Lambda - Mod$ and $H\Lambda - Mod^{\Sigma}$ with the model structure of [30, sect 4], with weak equivalences being weak equivalences between the underlying dg-modules and spectra respectively. By its definition, the Eilenberg-Mac Lane functor H preserves equivalences. Because we want invariants to take value in symmetric spectra, we compose the mixed complex functor with H , and the resulting functor is denote by

$$HH : dgCat_{/k} \longrightarrow H\Lambda - Mod^{\Sigma},$$

and called the Hochschild homology over k . It is still a localizing invariant. It is moreover a lax monoidal functor because Mix and H are lax monoidal.

In the following definition Hk is given the trivial action by the generator B of Λ .

Definition 3.2. Let $T \in \text{dgCat}/_k$.

- The cyclic homology symmetric spectrum of T is $\text{HC}(T) := \text{HH}(T) \wedge_{H\Lambda}^{\mathbb{L}} Hk$.
- The negative cyclic homology symmetric spectrum of T is $\text{HC}^-(T) := \mathbb{R}\underline{\text{Hom}}_{H\Lambda}(Hk, \text{HH}(T))$.
- The Hk -module $\text{HC}^-(T)$ is a module over the Hk -algebra $\mathbb{R}\underline{\text{Hom}}_{H\Lambda}(Hk, Hk) \simeq Hk[u]$ where $\deg(u) = -2$. We define the periodic cyclic homology symmetric spectrum of T by $\text{HP}(T) := \text{HC}^-(T) \wedge_{Hk[u]}^h Hk[u, u^{-1}]$.

These are functors

$$\text{HC}^- : \text{dgCat}/_k \longrightarrow Hk[u] - \text{Mod}^{\Sigma}$$

$$\text{HP} : \text{dgCat}/_k \longrightarrow Hk[u, u^{-1}] - \text{Mod}^{\Sigma}$$

There is a natural morphism of $Hk[u]$ -modules $\text{HC}^- \longrightarrow \text{HP}$, given by the unit in $k[u, u^{-1}]$.

If anything is specified, cyclic homology refers to cyclic homology over the base ring k .

We set some notations. If $\mathbf{E} : \text{dgCat}/_k \longrightarrow V$ is a functor, then we define a functor

$$\underline{\mathbf{E}} : \text{dgCat}/_k \longrightarrow \text{Pr}(\text{Aff}/_k, V)$$

by $\mathbf{E}(T)(A) = \mathbf{E}(T, A) := E(T \otimes_k^{\mathbb{L}} A)$. We will use this notation for all classical invariants of dg-categories like $\mathbf{E} = \mathbf{K}, \tilde{\mathbf{K}}, \text{HH}, \text{HC}, \text{HC}^-, \text{HP}$.

3.2 The algebraic Chern character as a \mathbf{K} -linear map

At this point we want to use Tabuada's category of "non-commutative motives" ([34], [4]), Tabuada–Cisinski's Theorem of corepresentability of \mathbf{K} -theory ([4, Thm 7.16]) in order to construct the algebraic Chern character in a $\mathbf{K}(k)$ -linear version, where $\mathbf{K}(k)$ is the presheaf of ring spectra of \mathbf{K} -theory of algebras. This is of great interest to us because we want to be able to invert the Bott element and thus to consider \mathbf{K} -theory and cyclic homology as modules over $\mathbf{K}(k)$. We recall the basic ideas to construct the model category $\mathbb{M}_{\text{loc}}(k)$, which is the value of the derivator $\mathcal{M}_{\text{loc}}(k)$ of Tabuada at the point. Then we will state a truncated version of the universal property of $\mathcal{M}_{\text{loc}}(k)$ which is sufficient for our purpose.

In fact there exists a variation of (a model of) $\mathcal{M}_{\text{loc}}(k)$ "with value in presheaves of spectra over $\text{Aff}/_k$ " rather than merely spectra themselves. We will describe this variation. Recall that $Sp^{\Sigma}(k)$ denotes the model category of presheaves of symmetric spectra over $\text{Aff}/_k$. The model category $\mathbb{M}_{\text{loc}}(k)$ can be obtained as a Bousfield localization of the projective model category $\text{Pr}(\text{dgCat}/_k^{fp}, Sp^{\Sigma}(k))$ of presheaves on homotopically finitely presented dg-categories with values in $Sp^{\Sigma}(k)$. The Bousfield localization consists to force invariants to commute with filtered homotopy colimits, send derived Morita equivalences to isomorphisms, and send exact sequences to distinguished triangles. The model category $\mathbb{M}_{\text{loc}}(k)$ is then a $Sp^{\Sigma}(k)$ -model category with $Sp^{\Sigma}(k)$ -enriched Hom denoted by $\underline{\text{Hom}}_{\mathbb{M}_{\text{loc}}(k)}$ with a derived version $\mathbb{R}\underline{\text{Hom}}_{\mathbb{M}_{\text{loc}}(k)}$, and by [3] it is also a monoidal model category in the sense of [17]. The monoidal structure is denoted by \wedge . This $\mathbb{M}_{\text{loc}}(k)$ is equipped with a functor

$$\underline{h} : \text{dgCat}/_k \longrightarrow \mathbb{M}_{\text{loc}}(k)$$

which is given by the (homotopy) Yoneda embedding $\underline{h}_T(T')(A) = \text{Hom}_{\text{dgCat}/_k}(T', T \otimes_k^{\mathbb{L}} A)$.

Proposition 3.3. The functor $\underline{h} : \text{dgCat}/_k \longrightarrow \mathbb{M}_{\text{loc}}(k)$

1. takes cofibrant values in $\mathbb{M}_{\text{loc}}(k)$,

2. commutes with filtered homotopy colimits,
3. sends derived Morita equivalences to weak equivalences,
4. sends exact sequences to distinguished triangles,

and is universal with respect to these properties, i.e. for every $Sp^\Sigma(k)$ -model category V , the induced functor

$$\underline{h}^* : \text{Fun}_!(\mathbb{M}_{loc}(k), V) \longrightarrow \text{Fun}_*(\text{dgCat}/k, V),$$

is an equivalence of categories, where $\underline{\text{Hom}}_!$ is the category of $Sp^\Sigma(k)$ -enriched left Quillen functors and $\underline{\text{Hom}}_*$ is the category of localizing invariants, ie functors which satisfies properties 1 to 4 above.

This proposition can be applied to the localizing invariant $\underline{\text{HH}}$ to obtain a $Sp^\Sigma(k)$ -enriched lax monoidal left Quillen functor still denoted by

$$\underline{\text{HH}} : \mathbb{M}_{loc}(k) \longrightarrow \text{Pr}(\text{Aff}/k, H\Lambda - \text{Mod}^\Sigma).$$

Theorem 3.4. (Tabuada-Cisinski) *There exists a canonical isomorphism in $Ho(\text{Fun}(\text{dgCat}/k, Sp^\Sigma(k)))$, $\mathbb{R}\underline{\text{Hom}}_{\mathbb{M}_{loc}(k)}(\underline{k}, \underline{h}) \simeq \mathbf{K}$, where \underline{k} is the object of $\mathbb{M}_{loc}(k)$ defined by $\underline{k}(T') = \text{Hom}_{\text{dgCat}/k}(T', k)$ and constant as a presheaf over Aff/k .*

The unit of the monoidal model category $\mathbb{M}_{loc}(k)$ is the object \underline{k} . Consider now the object \underline{h}_k of $\mathbb{M}_{loc}(k)$. It is given by $\underline{h}_k(T')(A) = \text{Map}_{\text{dgMor}/k}(T', A)$ for any $T' \in \text{dgCat}/k$ and any $A \in \text{CAlg}/k$. The object \underline{h}_k has the structure of a monoid in $\mathbb{M}_{loc}(k)$ because by definition of the monoidal structure in $\mathbb{M}_{loc}(k)$ we have $\underline{h}_k \wedge \underline{h}_k \simeq \underline{h}_{k \otimes_k k} \simeq \underline{h}_k$. We have an \underline{h}_k -module \underline{h}_T for every $T \in \text{dgCat}/k$, with action defined by the natural isomorphisms $\underline{h}_T \wedge \underline{h}_k \simeq \underline{h}_{T \otimes_k k} \simeq \underline{h}_T$.

Let's sum up the situation. We have the following objects

- A monoidal model category $\mathcal{V} = Sp^\Sigma(k) = \text{Pr}(\text{Aff}/k, Sp^\Sigma)$.
- Two \mathcal{V} -enriched monoidal model categories $\mathcal{M} = \mathbb{M}_{loc}(k)$ and $\mathcal{N} = \text{Pr}(\text{Aff}/k, H\Lambda - \text{Mod}^\Sigma)$. The units are denoted by $\mathbb{1}_{\mathcal{M}}$ and $\mathbb{1}_{\mathcal{N}}$ respectively. These are cofibrant objects of \mathcal{M} and \mathcal{N} respectively. Thus we have two monoidal functors $\mathcal{V} \longrightarrow \mathcal{M}$ and $\mathcal{V} \longrightarrow \mathcal{N}$ given by the product with the unit. Therefore their right adjoints $\underline{\text{Hom}}(\mathbb{1}, -)$ (\mathcal{V} -enriched Hom) are lax monoidal and sends monoids to monoids.
- A \mathcal{V} -enriched lax monoidal left Quillen functor $F : \mathcal{M} \longrightarrow \mathcal{N}$ given by $F = \underline{\text{HH}}$. (In our case $F(\mathbb{1}_{\mathcal{M}})$ is isomorphic to $\mathbb{1}_{\mathcal{N}}$).
- A (cofibrant) monoid $a = \underline{h}_k$ in \mathcal{M} and a (cofibrant) a -module m given by $m = \underline{h}_T$.
- Two maps in \mathcal{V} given by the functoriality of $\mathbb{L}F$:

$$t : \mathbb{R}\underline{\text{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, a) \longrightarrow \mathbb{R}\underline{\text{Hom}}_{\mathcal{N}}(F(\mathbb{1}_{\mathcal{M}}), F(a)),$$

$$u : \mathbb{R}\underline{\text{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, m) \longrightarrow \mathbb{R}\underline{\text{Hom}}_{\mathcal{N}}(F(\mathbb{1}_{\mathcal{M}}), F(m)),$$

where $\mathbb{R}\underline{\text{Hom}}_{\mathcal{M}}$ and $\mathbb{R}\underline{\text{Hom}}_{\mathcal{N}}$ are the derived $Ho(\mathcal{V})$ -enriched Homs.

By the existence of a model structure on categories of monoids and modules (see [30]), we can conclude the following

- The objects $\mathbb{R}\underline{\text{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, a)$ and $\mathbb{R}\underline{\text{Hom}}_{\mathcal{N}}(F(\mathbb{1}_{\mathcal{M}}), F(a))$ are monoids in \mathcal{V} and the map t is morphism of monoids.

- The object $\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, m)$ is a $\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, a)$ -module in \mathcal{V} .
- The object $\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{N}}(F(\mathbb{1}_{\mathcal{M}}), F(m))$ is a $\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{N}}(F(\mathbb{1}_{\mathcal{M}}), F(a))$ -module in \mathcal{V} and thus a $\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, a)$ -module in \mathcal{V} via t .
- The map u is a morphism of $\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, a)$ -modules in \mathcal{V} .

If we apply to our situation we obtain

- a presheaf of ring spectra $\underline{\mathbf{K}}(k) \simeq \mathbb{R}\underline{\mathrm{Hom}}_{\mathbb{M}_{loc}(k)}(k, h_k)$,
- a presheaf of $\underline{\mathbf{K}}(k)$ -modules $\underline{\mathbf{K}}(T) \simeq \mathbb{R}\underline{\mathrm{Hom}}_{\mathbb{M}_{loc}(k)}(k, h_T)$,
- a presheaf of $\underline{\mathbf{K}}(k)$ -modules $\mathbb{R}\underline{\mathrm{Hom}}(\underline{\mathrm{HH}}(k), \underline{\mathrm{HH}}(h_T)) \simeq \mathbb{R}\underline{\mathrm{Hom}}(k, \underline{\mathrm{HH}}(T/k)) \simeq \underline{\mathrm{HC}}^-(T)$, (where the $\mathbb{R}\underline{\mathrm{Hom}}$ is relative to the category $Pr(Aff_k, H\Lambda - Mod^{\Sigma})$),

and the following definition.

Definition 3.5. *The algebraic Chern character map is the map of $\underline{\mathbf{K}}(k)$ -modules defined by the functoriality of $\mathbb{L}\underline{\mathrm{HH}}(-/k)$:*

$$\underline{\mathbf{K}}(T) \simeq \mathbb{R}\underline{\mathrm{Hom}}_{\mathbb{M}_{loc}(k)}(k, h_T) \longrightarrow \mathbb{R}\underline{\mathrm{Hom}}_{Pr(Aff_k, H\Lambda - Mod^{\Sigma})}(\underline{\mathrm{HH}}(k), \underline{\mathrm{HH}}(h_T)) \simeq \underline{\mathrm{HC}}^-(T).$$

It gives a well defined map in $Ho(Fun(dgCat_k, \underline{\mathbf{K}}(k) - Mod^{\Sigma}))$,

$$ch : \underline{\mathbf{K}} \longrightarrow \underline{\mathrm{HC}}^-.$$

Remark 3.6. There exists an "additive version" $\mathbb{M}_a(k)$ of the model category $\mathbb{M}_{loc}(k)$ where we replace exact sequences of dg-categories with *split* exact sequences. Now the *connective* K-theory is corepresentable as a functor from $\mathbb{M}_a(k)$ to $Sp^{\Sigma}(k)$, and the same construction we performed gives a map

$$ch_c : \tilde{\mathbf{K}} \longrightarrow \underline{\mathrm{HC}}^-$$

in $Ho(Fun(dgCat_k, \tilde{\mathbf{K}}(k) - Mod^{\Sigma}))$, which is a connective version of the Chern character. Moreover, by construction the composite $\tilde{\mathbf{K}} \longrightarrow \underline{\mathbf{K}} \xrightarrow{ch} \underline{\mathrm{HC}}^-$ is isomorphic to ch_c .

Remark 3.7. By definition, the Chern character of Definition 3.5 agrees with Tabuada's Chern character. Moreover it has been explained to me by Bertrand Toën that in the case of $T = L_{pe}(X)$ the dg-category of perfect complexes on a complex algebraic variety X (eventually smooth quasi-projective over \mathbb{C}), this Chern character agrees with the classical Chern character from (higher) algebraic K-theory to De Rham cohomology as defined for example in the work of Gillet [14]. We hope to have the opportunity to give more details about this fact in future work.

4 Topological K-theory and its Chern character

4.1 Semi-topological and Topological K-theory

Definition 4.1. *The non-connective semi-topological K-theory or just semi-topological K-theory (resp. the connective semi-topological K-theory) of a \mathbb{C} -dg-category T is*

$$\mathbf{K}^{st}(T) := \mathbb{L}|\underline{\mathbf{K}}(T)| \quad (\text{resp. } \tilde{\mathbf{K}}^{st}(T) := \mathbb{L}|\tilde{\mathbf{K}}(T)|).$$

This defines functors

$$\mathbf{K}^{st}, \tilde{\mathbf{K}}^{st} : dgCat_{\mathbb{C}} \longrightarrow Sp^{\Sigma}.$$

The main motivations for the previous definition are the following two theorems whose proofs will be given below. Let \mathbf{bu} denotes the connective spectrum of complex topological K-theory. A particular model of it as a symmetric spectrum will be given.

Theorem 4.2. *There exists a canonical isomorphism in $Ho(Sp^\Sigma)$*

$$\tilde{\mathbf{K}}^{\text{st}}(\mathbb{C}) \simeq \mathbf{bu}.$$

Theorem 4.3. *For any smooth commutative algebra $B \in \text{CAlg}_{/\mathbb{C}}$, the canonical map of symmetric spectra $\tilde{\mathbf{K}}^{\text{st}}(B) \longrightarrow \mathbf{K}^{\text{st}}(B)$ is a weak equivalence. In particular by 4.2 there is an isomorphism $\mathbf{K}^{\text{st}}(\mathbb{C}) \simeq \mathbf{bu}$ in $Ho(Sp^\Sigma)$.*

Remark 4.4. Recall from Remark 3.6 that for every \mathbb{C} -dg-category T , the presheaf of spectra $\mathbf{K}(T)$ (resp. $\tilde{\mathbf{K}}(T)$) is a $\mathbf{K}(\mathbb{C})$ -module (resp. a $\tilde{\mathbf{K}}(\mathbb{C})$ -module). Thus, because the geometric realization functor $|-|$ commutes with smash products, the spectrum $\mathbf{K}^{\text{st}}(T)$ (resp. $\tilde{\mathbf{K}}^{\text{st}}(T)$) is a module over the ring spectrum $\mathbf{K}^{\text{st}}(\mathbb{C})$ (resp. over $\tilde{\mathbf{K}}^{\text{st}}(\mathbb{C})$). By abuse of notation, we will write \mathbf{bu} for the spectrum $\mathbf{K}^{\text{st}}(\mathbb{C}) \simeq \tilde{\mathbf{K}}^{\text{st}}(\mathbb{C})$ and consider it as a symmetric ring spectrum with the ring structure of $\mathbf{K}^{\text{st}}(\mathbb{C})$. This raises the question whether the ring structure on $\mathbf{K}^{\text{st}}(\mathbb{C})$ corresponds to the usual ring structure on \mathbf{bu} , with product coming from tensor product of topological vector bundles. The answer is yes and we hope we have the opportunity to give details about this in the work in progress [1].

The functors \mathbf{K}^{st} and $\tilde{\mathbf{K}}^{\text{st}}$ can then be lifted to functors

$$\mathbf{K}^{\text{st}}, \tilde{\mathbf{K}}^{\text{st}} : \text{dgCat}_{/\mathbb{C}} \longrightarrow \mathbf{bu} - \text{Mod}^\Sigma.$$

◇

We choose a Bott element $\beta \in \pi_2 \mathbf{bu}$. We denote by \mathbf{BU} the symmetric ring spectrum $\mathbf{bu}[\beta^{-1}]$, the localization of the symmetric ring spectrum \mathbf{bu} with respect to β . Recall that for every dg-category T , the spectra $\mathbf{K}^{\text{st}}(T)$ and $\tilde{\mathbf{K}}^{\text{st}}(T)$ are \mathbf{bu} -modules.

Definition 4.5. *Let $T \in \text{dgCat}_{/\mathbb{C}}$. The topological K-theory symmetric spectrum of T (resp. the connective topological K-theory symmetric spectrum of T) is by definition*

$$\mathbf{K}^{\text{top}}(T) := \mathbf{K}^{\text{st}}(T)[\beta^{-1}], \quad (\text{resp. } \tilde{\mathbf{K}}^{\text{top}}(T) := \tilde{\mathbf{K}}^{\text{st}}(T)[\beta^{-1}]),$$

the localization of the \mathbf{bu} -module $\mathbf{K}^{\text{st}}(T)$ (resp. $\tilde{\mathbf{K}}^{\text{st}}(T)$) with respect to the Bott element β , in the category Sp^Σ . This defines functors

$$\mathbf{K}^{\text{top}}, \tilde{\mathbf{K}}^{\text{top}} : \text{dgCat}_{/\mathbb{C}} \longrightarrow \mathbf{BU} - \text{Mod}^\Sigma.$$

4.2 Connective theory, \mathbf{bu} and the moduli stack of perfect dg-modules

For any algebra $A \in \text{CAlg}_{/\mathbb{C}}$, let $\text{Vect}_{\mathbb{C}}(A)$ denotes the category of projective A -modules of finite type. Using tensor product, this is lax functorial in $A \in \text{CAlg}_{/\mathbb{C}}$. Let $\text{Vect}_{\mathbb{C}}$ denotes the canonical strictification of this lax functor. The direct sum of modules gives a Γ -simplicial presheaf $\text{Vect}_{\mathbb{C}}^\bullet$ whose level 1 is the the classifying presheaf of objects in $\text{Vect}_{\mathbb{C}}$. More precisely in term of the construction B_\bullet of [1] which goes back to Segal, we have $\text{Vect}_{\mathbb{C}}^\bullet = \text{Nw} B_\bullet \text{Vect}_{\mathbb{C}}$, where Nw here means the nerve of isomorphisms. One particular model for the connective complex topological K-theory spectrum \mathbf{bu} is :

$$\mathbf{bu} := \mathcal{B}\mathbb{L}[\text{Vect}_{\mathbb{C}}^\bullet]^+.$$

The reason for this is that there exists an étale local weak equivalence of Γ -simplicial presheaves,

$$\text{Vect}_{\mathbb{C}}^\bullet \simeq \bigsqcup_{n \geq 0} BGL_n,$$

where Gl_n is the presheaf of groups such that $Gl_n(A)$ is the standard linear group considered as a discrete simplicial set, and the sum is given by the block sum of matrices. Therefore by 2.18 we have $\mathbb{L}|Vect_{\mathbb{C}}^{\bullet}|^+ \simeq (\bigsqcup_{n \geq 0} \mathbb{L}|BGl_n|)^+ \simeq (\bigsqcup_{n \geq 0} B\mathbb{L}|Gl_n|)^+ \simeq (\bigsqcup_{n \geq 0} B|Gl_n|)^+ \simeq (\bigsqcup_{n \geq 0} BGl_n(\mathbb{C}))^+$, where $Gl_n(\mathbb{C})$ is now considered as a topological group. But $Gl_n(\mathbb{C})$ has the same homotopy type as $U_n(\mathbb{C})$, hence $\mathbb{L}|Vect_{\mathbb{C}}^{\bullet}|^+ \simeq (\bigsqcup_{n \geq 0} BU_n(\mathbb{C}))^+$. But by [12, Appendice Q] $(\bigsqcup_{n \geq 0} BU_n(\mathbb{C}))^+ \simeq BU_{\infty} \times \mathbb{Z}$, where BU_{∞} is the colimit of the $BU_n(\mathbb{C})$ with the standard inclusions. The structure of Γ -object is still given by the block sum of matrices and the usual sum for \mathbb{Z} . In consequence by Theorem 2.6, we have $\mathcal{B}\mathbb{L}|Vect_{\mathbb{C}}^{\bullet}|^+ \simeq \mathcal{B}(BU_{\infty} \times \mathbb{Z})$ which is the usual definition of the spectrum **bu**.

Proof. (of Theorem 4.2)

We have a chain of weak equivalences in Sp^{Σ} :

$$\begin{aligned} \tilde{\mathbf{K}}^{\text{st}}(\mathbb{C}) &= \mathbb{L}|\tilde{\mathbf{K}}(\mathbb{C}, -)| \\ &\simeq \mathbb{L}|\tilde{\mathbf{K}}(Vect_{\mathbb{C}})| && \text{(from Remark 2.9)} \\ &= \mathbb{L}|\mathcal{B}K^{\Gamma}(Vect_{\mathbb{C}})| && \text{(by definition of the K-theory spectrum)} \\ &\simeq \mathcal{B}\mathbb{L}|K^{\Gamma}(Vect_{\mathbb{C}})| && \text{(from Proposition 2.25)} \end{aligned}$$

For any Waldhausen category C , there is a map of simplicial sets

$$NwC \longrightarrow K(C) = \Omega|NwS_{\bullet}C|,$$

which is the adjoint of the map obtained by the inclusion of the 1-skeleton in the realization

$$|NwS_{\bullet}C| \simeq coeq(\bigsqcup_n NwS_n C \times \Delta^n \rightrightarrows \dots).$$

The same map is obtained from $B_{\bullet}Vect_{\mathbb{C}}$ which is not a Waldhausen category but a Γ -presheaf of Waldhausen categories. Hence we have a map

$$\sigma : Vect_{\mathbb{C}}^{\bullet} \longrightarrow K^{\Gamma}(Vect_{\mathbb{C}}) = K(B_{\bullet}Vect_{\mathbb{C}}).$$

We claim that this map induces a weak equivalence on group completion, which is equivalent to have a weak equivalence

$$\sigma^+ : (Vect_{\mathbb{C}}^{\bullet})^+ \longrightarrow K^{\Gamma}(Vect_{\mathbb{C}})$$

because $K^{\Gamma}(Vect_{\mathbb{C}})$ is already group complete. Because these Γ -objects are special, it suffices to have a weak equivalence on the level 1 :

$$\sigma_1^+ : (Vect_{\mathbb{C}}^{\bullet})_1^+ \longrightarrow K^{\Gamma}(Vect_{\mathbb{C}})_1 \simeq K(Vect_{\mathbb{C}}).$$

But this last map is the level 1 of another weak equivalence of Δ -spaces. Indeed, let C be any Waldhausen category where every cofibration is split. Then the map

$$\begin{aligned} \lambda : B_{\bullet}C &\longrightarrow S_{\bullet}C \\ (a_1, \dots, a_n) &\longmapsto (a_1 \hookrightarrow a_1 \oplus a_2 \hookrightarrow \dots \hookrightarrow a_1 \oplus \dots \oplus a_n) \end{aligned}$$

is a strict equivalence in $\Delta - WCat$ (a quasi inverse is given by taking successive quotients of any sequence). We prove below that λ is indeed a simplicial map. Because the presheaf of Waldhausen categories $Vect_{\mathbb{C}}$ has only split cofibrations, we obtain a weak equivalence still denoted by

$$\lambda : Vect_{\mathbb{C}}^{\bullet} \xrightarrow{\sim} \mathcal{K}_{\bullet}(Vect_{\mathbb{C}}) := NwS_{\bullet}Vect_{\mathbb{C}},$$

in $\Delta - SSet$. Hence a weak equivalence on group completion (and monoidification for the target, because it is not even special)

$$\lambda^+ : (Vect_{\mathbb{C}}^{\bullet})^+ \xrightarrow{\sim} (mon \mathcal{K}_{\bullet}(Vect_{\mathbb{C}}))^+.$$

But the level 1 of $(mon \mathcal{K}_{\bullet}(Vect_{\mathbb{C}}))^+$ is equivalent to $\Omega|NwS_{\bullet}Vect_{\mathbb{C}}| = K(Vect_{\mathbb{C}})$ by Segal's Theorem [31, prop 1.5] and the map λ_1^+ is equivalent to the map σ_1^+ . We then have that σ_1^+ and hence σ^+ are weak equivalences. We obtain isomorphisms in $Ho(Sp^{\Sigma})$:

$$\begin{aligned} \tilde{\mathbf{K}}^{st}(\mathbb{C}) &\simeq \mathcal{B}\mathbb{L}|K^{\Gamma}(Vect_{\mathbb{C}})| \\ &\simeq \mathcal{B}\mathbb{L}|(Vect_{\mathbb{C}}^{\bullet})^+| \\ &\simeq \mathcal{B}\mathbb{L}|Vect_{\mathbb{C}}^{\bullet}|^+ && \text{(from Proposition 2.24)} \\ &= \mathbf{bu} \end{aligned}$$

□

Let C be any Waldhausen category and let's return to the map

$$\begin{aligned} \lambda : B_{\bullet}C &\longrightarrow S_{\bullet}C \\ (a_1, \dots, a_n) &\longmapsto (a_1 \hookrightarrow a_1 \oplus a_2 \hookrightarrow \dots \hookrightarrow a_1 \oplus \dots \oplus a_n) \end{aligned}$$

Then it defines a map in $\Delta - WCat$. Indeed, this verification in the general case (ie at any simplicial level) is similar to what happens between simplicial dimensions 1 and 2. To see this on the levels 1 and 2, let's write the effect of the low degree face and degeneracy maps. For any $a, b \in B_1C$, and any cofibration $x \hookrightarrow y \in S_2C$,

$B_{\bullet}C$	$S_{\bullet}C$
$d_0(a, b) = b$	$d_0(x \hookrightarrow y) = y/x$
$d_1(a, b) = a \oplus b$	$d_1(x \hookrightarrow y) = y$
$d_2(a, b) = a$	$d_2(x \hookrightarrow y) = x$
$s_0(a) = (0, a)$	$s_0(a) = (0 \hookrightarrow a)$
$s_1(a) = (a, 0)$	$s_1(a) = id_a$

Then one can see that

$$\begin{aligned} \lambda(d_0(a, b)) &= \lambda(b) = b = d_0(a \hookrightarrow a \oplus b) = d_0(\lambda(a, b)), \\ \lambda(d_1(a, b)) &= \lambda(a \oplus b) = a \oplus b = d_1(a \hookrightarrow a \oplus b) = d_1(\lambda(a, b)), \\ \lambda(d_2(a, b)) &= \lambda(a) = a = d_2(a \hookrightarrow a \oplus b) = d_2(\lambda(a, b)), \\ \lambda(s_0(a)) &= \lambda(0, a) = (0 \hookrightarrow a) = s_0(a) = s_0(\lambda(a)), \\ \lambda(s_1(a)) &= \lambda(a, 0) = id_a = s_1(a) = s_1(\lambda(a)). \end{aligned}$$

Let now $T \in dgCat_{/\mathbb{C}}$ and let's apply it to the presheaf of categories $Perf(T, -)$, and take the nerve of weak equivalences. We set the following notations

$$\mathcal{M}_{\bullet}^T := NwB_{\bullet}Perf(T, -).$$

$$\mathcal{K}_{\bullet}^T := \mathcal{K}_{\bullet}(Perf(T, -)) = NwS_{\bullet}Perf(T, -).$$

The simplicial presheaf $\mathcal{M}^T := \mathcal{M}_1^T = NwPerf(T, -)$ is the moduli stack of perfect T^{op} -dg-modules, which is the "dual" of the moduli stack \mathcal{M}_T of pseudo perfect T^{op} -dg-modules, main object of study in [41].

We then have a map in $\Delta - \text{SPR}(\mathbb{C})^{\text{ét}, \mathbf{A}^1}$:

$$\lambda : \mathcal{M}_\bullet^T \longrightarrow \mathcal{K}_\bullet^T$$

The following result is central in our study of topological K-theory. It will be used below to rely our $\tilde{\mathbf{K}}^{\text{st}}(T)$ to the Γ -moduli stack \mathcal{M}_\bullet^T . A formula which was taken as a definition in previous surveys on the subject (see [22, 20]) but which captures only the connective part of the spectrum.

Proposition 4.6. *The map $\lambda : \mathcal{M}_\bullet^T \longrightarrow \mathcal{K}_\bullet^T$ is a levelwise \mathbf{A}^1 -equivalence in $\Delta - \text{SPR}(\mathbb{C})^{\text{ét}, \mathbf{A}^1}$.*

As the proof is not really enlightening for the purpose of this section, it is postponed to next one.

Proposition 4.7. *Let $T \in \text{dgCat}/\mathbb{C}$. The special Γ -space $|\mathcal{M}_\bullet^T|$ is very special.*

Proof. It is equivalent to prove the claim using the topological geometric realization $|-|^{top}$ instead of the simplicial geometric realization. By Lemma 2.3, our statement is equivalent to the statement that the monoid $\pi_0|\mathcal{M}^T|^{top}$ is a group. Let $[E] \in \pi_0|\mathcal{M}^T|^{top}$ be the connected component of a class of a dg-module E . Then we need to find an inverse with respect to the direct sum to $[E]$. We claim the inverse is given by the shifted dg-module $[E[1]]$. To verify the claim one need to define a path

$$\gamma : [0, 1] \longrightarrow |\mathcal{M}^T|^{top}$$

with $\gamma(0) = E \oplus E[1]$ and $\gamma(1) = 0$, where $[0, 1]$ is the real unit interval. To define such a path it is sufficient to define a map of simplicial presheaves

$$\delta : \mathbf{A}^1 \longrightarrow \mathcal{M}^T$$

with $\delta(0) \simeq E \oplus E[1]$ and $\delta(1) \simeq 0$ where the maps \simeq are quasi-isomorphisms. Because then, taking the functor $|-|$, we obtain a path γ using any homotopy equivalence $|\mathbf{A}^1| \simeq [0, 1]$ and paths given by the quasi-isomorphisms. The map δ is defined on an algebra $A \in \text{Alg}/\mathbb{C}$ by

$$\delta_A(f) := \text{cone}(E \xrightarrow{\times f} E).$$

Then one has $\delta_A(0) = \text{cone}(0 : E \rightarrow E) = E \oplus E[1]$ with the sum differential, and $\delta_A(1) = \text{cone}(\text{id} : E \rightarrow E)$ which is canonically quasi-isomorphic to the zero A -dg-module. \square

Proposition 4.8. *Let $T \in \text{dgCat}/\mathbb{C}$. The special Γ -space $\mathbb{L}|\mathcal{M}_\bullet^T|$ is very special.*

Proof. We use the formula 2.20 and the proof of Lemma 4.7. Indeed the set $\pi_0\mathbb{L}|\mathcal{M}_\bullet^T|$ is isomorphic to the set $\pi_0\mathcal{M}^T(\mathbb{C})/\sim$ where two connected components of $T^{op} \otimes \mathbb{C}$ -dg-modules $[E]$ and $[E']$ are equivalent if they are linked by a connected algebraic curve in $\mathcal{M}^T(\mathbb{C})$ according to 2.20. But given any $T^{op} \otimes \mathbb{C}$ -dg-module E , in the proof of Proposition 4.7 we defined a morphism $\delta : \mathbf{A}^1 \longrightarrow \mathcal{M}^T$ such that $\delta(0) = E \oplus E[1]$ and $\delta(1) = 0$ in $\pi_0\mathcal{M}^T(\mathbb{C})$. Thus in $\pi_0\mathbb{L}|\mathcal{M}_\bullet^T|$ we have the identity $[E \oplus E[1]] = [0]$, which proves it is a group. \square

Corollary 4.9. *Let $T \in \text{dgCat}/\mathbb{C}$. There is a canonical isomorphism in $Ho(\text{Sp}^\Sigma)$:*

$$\tilde{\mathbf{K}}^{\text{st}}(T) \simeq \mathcal{B}\mathbb{L}|\mathcal{M}_\bullet^T|.$$

Proof. It is the same arguments as in the proof of Theorem 4.2, except that the cofibrations are not split anymore. So this fact is somewhat replaced by Proposition 4.6. We have a chain of weak equivalences

$$\begin{aligned} \tilde{\mathbf{K}}^{\text{st}}(T) &:= \mathbb{L}|\tilde{\mathbf{K}}(T, -)| \\ &= \mathbb{L}|\mathcal{B}K^\Gamma(T, -)| && \text{(by definition of the K-theory spectrum)} \\ &\simeq \mathcal{B}\mathbb{L}|K^\Gamma(T, -)| && \text{(from Proposition 2.25)} \end{aligned}$$

As in the proof of 4.2 we have a map

$$\sigma : \mathcal{M}_\bullet^T \longrightarrow K^\Gamma(T, -),$$

and we claim σ induces an isomorphism on derived geometric realization. Because these Γ -objects are special, it suffices to have an equivalence on the level 1, hence for $\mathbb{L}|\sigma|_1$ to be a weak equivalence. The map λ is an \mathbf{A}^1 -equivalence by 4.6, hence the map $\mathbb{L}|\lambda|_1^+$ is a weak equivalence, and in particular on the level 1. We have a commutative diagram in $Ho(SSet)$

$$\begin{array}{ccccc} \mathbb{L}|\mathcal{M}_\bullet^T|_1 & \xrightarrow{\mathbb{L}|\sigma|_1} & \mathbb{L}|K^\Gamma(T, -)|_1 & \xrightarrow{\sim} & \mathbb{L}|K(T, -)| \\ \downarrow \wr & & & \nearrow \sim & \\ \mathbb{L}|\mathcal{M}_\bullet^T|_1^+ & \xrightarrow{\mathbb{L}|\lambda|_1^+} & \mathbb{L}|\mathcal{K}_\bullet^T|_1^+ & & \end{array}$$

where all the maps are isomorphism except a priori $\mathbb{L}|\sigma|_1$, but we conclude it is. Therefore, $\mathbb{L}|\sigma|$ is a weak equivalence. We conclude the existence of an isomorphism $\tilde{\mathbf{K}}^{\text{st}}(T) \simeq \mathcal{B}\mathbb{L}|\mathcal{M}_\bullet^T|$ in $Ho(Sp^\Sigma)$. \square

4.3 Proof of Proposition 4.6

We introduce some notations first. Let $T \in dgCat/\mathbb{C}$. The simplicial presheaf \mathcal{K}_n^T classifies sequences of $n - 1$ cofibrations in $Perf(T, -)$. For every $n \geq 1$, let

$$M^{(n)} := Fun([n - 1], Perf(T, -))$$

denotes the presheaf of categories of sequences of $n - 1$ composable morphisms in $Perf(T, -)$, where a morphism between such sequences $a_1 \rightarrow \dots \rightarrow a_n$ and $b_1 \rightarrow \dots \rightarrow b_n$ is a given by commutative squares

$$\begin{array}{ccccccc} a_1 & \longrightarrow & a_2 & \longrightarrow & \dots & \longrightarrow & a_n \\ \downarrow & & \downarrow & & & & \downarrow \\ b_1 & \longrightarrow & b_2 & \longrightarrow & \dots & \longrightarrow & b_n \end{array}$$

One has $M^{(1)} = Perf(T, -)$. Then for every $n \geq 1$ and every $A \in CAlg/\mathbb{C}$ the category $M^{(n)}(A)$ is naturally endowed with a model structure, namely the projective model structure. Let $X^{(n)} := NwM^{(n)}$ be the simplicial presheaf classifying sequences of $n - 1$ composable morphisms in $Perf(T, -)$ up to quasi-isomorphisms. For every $n \geq 1$, there is a natural inclusion

$$\mathcal{K}_T^{(n)} \hookrightarrow X^{(n)}.$$

Because every morphisms in $Perf(T, A)$ can be factorized by a cofibration followed by weak equivalence, this last inclusion is a global weak equivalence in $SPr(\mathbb{C})$. Hence Proposition 4.6 is equivalent to the statement that the composite map

$$\alpha^* \mathcal{M}_n^T \longrightarrow X^{(n)}$$

is an \mathbf{A}^1 -equivalence for every $n \geq 1$. The advantage is that we now do not care if the maps considered are cofibrations or not.

We proceed by induction by reducing the assertion at level n to the assertion at levels $n - 1$ and 2. At level 1 : the two presheaves are equal $\mathcal{M}_1^T = X^{(1)} = NwPerf(T, -)$. At level 2 : the map $\lambda^{(2)}$ can be written on 0-simplices as

$$\lambda^{(2)}(a, b) = (a \rightarrow a \oplus b).$$

Then we define an explicit \mathbf{A}^1 -homotopy inverse to $\lambda^{(2)}$ which we denote by $\mu^{(2)}$ and define on 0-simplexes as

$$\mu^{(2)}(i : x \rightarrow y) := (x, \text{cone}(i)).$$

The map $\mu^{(2)}$ extends by functoriality to the whole nerve of weak equivalences in $\text{Perf}(T, -)$. By direct calculation we have

$$\mu^{(2)} \circ \lambda^{(2)}(a, b) = \mu^{(2)}(a \rightarrow a \oplus b) = (a, \text{cone}(a \rightarrow a \oplus b)) \simeq (a, b),$$

where the symbol \simeq is a canonical quasi-isomorphism. Hence $\mu^{(2)} \circ \lambda^{(2)} \simeq \text{id}$. In the other direction it is more complicated since

$$\lambda^{(2)} \circ \mu^{(2)}(i : x \rightarrow y) = \lambda^{(2)}(x, \text{cone}(i)) = (x \rightarrow x \oplus \text{cone}(i)).$$

We then write an \mathbf{A}^1 -homotopy $h : \mathbf{A}^1 \times X^{(1)} \rightarrow X^{(1)}$ as follows. It is defined on any algebra $A \in \text{CAlg}/\mathbb{C}$ and on 0-simplexes by

$$\begin{aligned} h_A : A \times X^{(1)}(A) &\rightarrow X^{(1)}(A) \\ (f, i : x \rightarrow y) &\mapsto (fi : x \rightarrow y). \end{aligned}$$

The map h is an \mathbf{A}^1 -homotopy between $\text{id}_{X^{(1)}}$ and the endomorphism Z of $X^{(1)}$ given by $Z(i : x \rightarrow y) = 0 : x \rightarrow y$. Nevertheless, what we look for is an \mathbf{A}^1 -homotopy between $\text{id}_{X^{(1)}}$ and $\lambda^{(2)} \circ \mu^{(2)}$. In fact the endomorphism Z is conjugated by a (global) auto-equivalence of $X^{(1)}$ to the endomorphism $\lambda^{(2)} \circ \mu^{(2)}$. The claimed auto-equivalence is defined on 0-simplexes by

$$\begin{aligned} t : X^{(1)} &\rightarrow X^{(1)} \\ (i : x \rightarrow y) &\mapsto (y \rightarrow \text{cone}(i)), \end{aligned}$$

and extends by functoriality to the whole nerve. The map t composed three times with itself gives the shift by 1, hence t is a weak equivalence. The inverse of t is given by

$$t^{-1}(i : x \rightarrow y) = \text{cocone}(i) \rightarrow x,$$

where the last morphism is the natural morphism given by the definition of the cocone. One has

$$\begin{aligned} tZt^{-1}(f) &= tZ(\text{cocone}(i) \rightarrow x) \\ &= t(0 : \text{cocone}(i) \rightarrow x) \\ &= x \rightarrow \text{cone}(0 : \text{cocone}(i) \rightarrow x). \end{aligned}$$

The dg-module $\text{cone}(0 : \text{cocone}(i) \rightarrow x)$ is naturally isomorphic to the dg-module $x \oplus \text{cone}(i)$ with the sum differential. Hence, one has $tZt^{-1} \simeq \lambda^{(2)} \circ \mu^{(2)}$. To sum up, the map h is an homotopy $\text{id}_{X^{(1)}} \Rightarrow Z$ but one has $tZt^{-1} \simeq \lambda^{(2)} \circ \mu^{(2)}$, which implies that $\lambda^{(2)}$ is an \mathbf{A}^1 -equivalence.

Let now $n \geq 2$. Recall from [38] the notion of fibered product for model categories. Then there is a functor

$$\begin{aligned} F : M^{(n)} &\rightarrow M^{(n-1)} \times_{M^{(1)}} M^{(2)} \\ (a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_n) &\mapsto ((a_1 \rightarrow \cdots \rightarrow a_{n-2} \rightarrow a_n), (a_{n-1}/a_{n-2} \rightarrow a_n/a_{n-2}), a_{n-1}/a_{n-2}, \text{id}, \text{id}) \end{aligned}$$

where $a_{n-2} \rightarrow a_n$ is the composite of $a_{n-2} \rightarrow a_{n-1} \rightarrow a_n$, and the notation "quotient" stands for the homotopy cofiber (or the cone) of a morphism.

Lemma 4.10. *With the notation above, the functor F satisfies the two conditions of [38, Lemma 4.2]. Hence the induced map*

$$q^{(n)} : X^{(n)} \longrightarrow X^{(n-1)} \underset{X^{(1)}}{\overset{h}{\times}} X^{(2)}$$

is a global weak equivalence in $SPr(\mathbb{C})$, (the homotopy fibered product being calculated in the global model structure).

Proof. We will not give an explicit proof because Toën proved in his paper (just after the proof of Lemma 4.2) that the map $q^{(3)}$ is a weak equivalence, and his proof can be directly generalized to sequences of arbitrary length, (we point out that there is a shift of indices between our indices and the paper [38]).

Nevertheless there are two important things to say. The first is that we do not work in a stable model category as in the setting of [38, Lemma 4.2] (eg all dg-modules over a dg-category) but rather with perfect objects in a stable model category. Hopefully, the proof in [38, Lemma 4.2] still makes sense and works for perfect objects. The second is that we do not even work with a category but with a presheaf of categories. But [38, Lemma 4.2] can be applied objectwise and we know that, in the global model structure, a square of simplicial presheaves is homotopy cartesian if and only if it is objectwise homotopy cartesian. \square

Therefore we end up with a square

$$\begin{array}{ccc} \mathcal{M}_n^T & \xrightarrow{\lambda^{(n)}} & X^{(n)} \\ \downarrow p & & \downarrow q^{(n)} \\ \mathcal{M}_{n-1}^T \times \mathcal{M}_2^T & \xrightarrow{\lambda^{(n-1)} \times \lambda^{(2)}} & X^{(n-1)} \underset{X^{(1)}}{\overset{h}{\times}} X^{(2)} \end{array}$$

where p is taken to be the map $p(a_1, \dots, a_n) = ((a_1, \dots, a_{n-2}, a_{n-1} \oplus a_n), (a_{n-1}, a_n))$. The map p is a global weak equivalence by definition. It is chosen in order for the square to homotopy commute. Indeed one can check it by direct calculation on 0-simplexes and this shows by functoriality that the square is homotopy commutative. The maps $\lambda^{(n-1)}$ and $\lambda^{(2)}$ are \mathbf{A}^1 -homotopy equivalences by induction hypothesis. Now because \mathbf{A}^1 -homotopy equivalences are stable by fibered product, the map $\lambda^{(n-1)} \times \lambda^{(2)}$ is an \mathbf{A}^1 -homotopy equivalence. We conclude by the two out of three property that the map $\lambda^{(n)}$ is an \mathbf{A}^1 -equivalence, which finishes the proof of 4.6.

4.4 Non-connective theory and proper hyperdescent in topology

We now give a proof of Theorem 4.3.

Proof. (of Theorem 4.3). The natural map of presheaves $\tilde{\mathbf{K}}(B, -) \longrightarrow \mathbf{K}(B, -)$ is an equivalence when restricted to smooth affine schemes (see for example [28, Remark 7]). Then we use the étale-proper topology on $Aff_{/\mathbb{C}}$ defined below. Every $X \in Aff_{/\mathbb{C}}$ admits an étale-proper covering by a smooth scheme (not in $Aff_{/\mathbb{C}}$ in general), and every simplicial presheaf that satisfies étale-proper hyperdescent satisfies étale descent and thus Zariski descent. This implies that the map $\tilde{\mathbf{K}}(B, -) \longrightarrow \mathbf{K}(B, -)$ is an étale-proper local equivalence. Then we use the fact proved below that the functor $\mathbb{L}|-| : Ho(Sp^{\Sigma}(\mathbb{C})) \longrightarrow Ho(Sp^{\Sigma})$ sends étale-proper local equivalences to weak equivalences. This proves our claim. \square

Definition 4.11. *The étale-proper topology on the category $Aff_{/\mathbb{C}}$ (étp for short) is by definition the weakest Grothendieck topology on $Aff_{/\mathbb{C}}$ such that étale surjective morphisms and proper surjective morphisms are coverings.*

Because of the resolution of singularities over \mathbb{C} , every scheme in Aff/\mathbb{C} admits an étale-proper covering by a smooth scheme. It remains to show that the geometric realization behaves well with respect to the étale-proper topology. We denote by $SPr(\mathbb{C})^{\acute{e}tp}$ the étale-proper local model category of simplicial presheaves, i.e. the left Bousfield localization of $SPr(\mathbb{C})$ with respect to the set of maps of the form

- $hocolim_{\Delta^{op}} Y_{\bullet} \rightarrow X$ for $Y_{\bullet} \rightarrow X$ an étale-proper hypercovering of a scheme $X \in Aff/\mathbb{C}$.

One defines in the same way the $\acute{e}tp$ -local model category of presheaves of symmetric spectra $Sp^{\Sigma}(\mathbb{C})^{\acute{e}tp}$.

Proposition 4.12. *If $Y_{\bullet} \rightarrow X$ is a proper hypercovering in Aff/\mathbb{C} then the induced map*

$$hocolim|Y_{\bullet}|^{top} \rightarrow |X|^{top}$$

is a weak equivalence of topological spaces. The geometric realization functor

$$|-|^{top} : SPr(\mathbb{C})^{\acute{e}tp} \rightarrow Top$$

is left Quillen, and the spectral realization

$$|-| : Sp^{\Sigma}(\mathbb{C})^{\acute{e}tp} \rightarrow Sp^{\Sigma}$$

is left Quillen.

Proof. The second statement follows from the first by general non sense about Bousfield localizations and from the fact that a simplicial presheaf has hyperdescent with respect to the topology $\acute{e}tp$ if and only if it has hyperdescent with respect both topology étale and proper.

Let's adopt the notation $A = |X|^{top}$ and $B_{\bullet} = |Y_{\bullet}|^{top}$. The map $B_{\bullet} \rightarrow A$ is a proper topological hypercovering, i.e. an hypercovering for the topology on Top with coverings being proper surjective continuous maps. In consequence it suffices to prove that for any proper topological hypercovering $B_{\bullet} \rightarrow A$ with spaces $(B_n)_n$ and A being sufficiently nice, then the map $hocolim_{\Delta^{op}} B_{\bullet} \rightarrow A$ is a weak equivalence. Here sufficiently nice is locally compact, Hausdorff and having the homotopy type of CW complexes. These conditions are satisfy by the geometric realization of any affine \mathbb{C} -scheme of finite type. Then the proof is complete by Proposition 4.13 below. \square

Proposition 4.13. *Let $B_{\bullet} \rightarrow A$ be a proper hypercovering of spaces with $(B_n)_n$ and A being locally compact Hausdorff topological spaces having the homotopy type of CW complexes, then the map*

$$hocolim_{\Delta^{op}} B_{\bullet} \rightarrow A$$

is a weak equivalence of spaces.

Proof. The proof will consist of several steps which mimic the proof by Dugger-Isaksen of the "open" version of our statement in [8, Theorem 4.3]. The difference is that covers here are not open covers but proper covers. We resume these steps now. First we reduce the statement to the case of bounded hypercoverings (in the sense of [7, definition 4.10]) using the same argument as in the proof of [8, Theorem 4.3]. Second we reduce to the case of a simpler class of bounded proper hypercoverings which are "nerves" of proper surjective morphisms using the same argument as in the proof [8, Lemma 4.2]. Third, we prove it for this class of hypercoverings using a proper base change theorem to reduce to the case of $A = \text{a point}$. Then a lemma purely from simplicial homotopy theory provides the result for the point. It's in the third step that we need these assumptions on the topological spaces, in order to perform a proper base change.

Let's introduce some terminology. Let \mathcal{C} be any complete and cocomplete category. For any $[n] \in \Delta$ and any simplicial object C_{\bullet} in \mathcal{C} we denote by $sk_n C$ its n -skeleton and $cosk_n C_{\bullet}$ its n -coskeleton. If

$\mathcal{C} = \text{Top}$, one has $(\text{cosk}_n C_\bullet)_i \simeq \text{Map}(\text{sk}_n \Delta^i, C_\bullet)$, where Map is a mapping space for $\text{Top}^{\Delta^{op}}$. There is an augmented version of these, if $C_\bullet \rightarrow D$ is an augmented simplicial object to a constant simplicial object D , then we denote by $\text{sk}_n^D C_\bullet$ and $\text{cosk}_n^D C_\bullet$ the the skeleton and coskeleton functor for the category $(\mathcal{C} \downarrow D)^{\Delta^{op}}$. We denote by

$$M_n C = \lim_{(\Delta^{op} \downarrow n) \setminus id} C_\bullet$$

the n th matching object of C_\bullet , where $(\Delta^{op} \downarrow n) \setminus id$ is the category of maps to $[n]$ in Δ^{op} minus the identity map of $[n]$. There is an augmented version of the matching object. If $C_\bullet \rightarrow D$ is an augmented simplicial object of \mathcal{C} into a constant simplicial object D , then one can compute the limit seeing C as a functor from $(\Delta^{op} \downarrow n) \setminus id$ to the category $\mathcal{C} \downarrow D$ of maps to D in \mathcal{C} . We denote it by $M_n^D C_\bullet$. There are natural maps $C_n \rightarrow M_n C_\bullet$ and $C_n \rightarrow M_n^D C_\bullet$. Suppose \mathcal{C} is endowed with a Grothendieck topology so that we can talk about hypercoverings, for us it will be Top with the proper topology. A hypercovering $C_\bullet \rightarrow D$ is called *bounded* if there exists an integer $N \geq 0$ such that the maps $C_n \rightarrow M_n^D C_\bullet$ are isomorphisms for all $n > N$. The minimum N with this property is called the *dimension* of the hypercovering. A hypercovering is bounded of dimension $\leq N$ if and only if the unit map $C_\bullet \rightarrow \text{cosk}_N^D C_\bullet$ is an isomorphism.

If $f : C \rightarrow D$ is a map in \mathcal{C} , one can see it as a map of constant simplicial objects of \mathcal{C} . Then we can take the 0-coskeleton $\text{cosk}_0^D C \rightarrow D$. This augmented simplicial object is called the nerve of f . We have $(\text{cosk}_0^D C)_i = C \times_D \cdots \times_D C$, $n+1$ times. The faces and degeneracies are the projections and diagonals respectively. If $f : C \rightarrow D$ is a covering in \mathcal{C} then the nerve of f is an hypercovering of D of dimension 0, and these are the only hypercoverings of dimension 0.

To reduce to the case of bounded hypercoverings, we observe that for any $k \geq 0$, the hypercovering $\text{cosk}_{k+1}^A B_\bullet$ is bounded and that the unit map $B_\bullet \rightarrow \text{cosk}_{k+1}^A B_\bullet$ is an isomorphism on $(k+1)$ -skeleton. By Lemma 4.14 below, this implies that the top map in the diagram

$$\begin{array}{ccc} \text{hocolim}_{\Delta^{op}} B_\bullet & \longrightarrow & \text{hocolim}_{\Delta^{op}} \text{cosk}_{k+1}^A B_\bullet \\ & \searrow & \downarrow \\ & & A \end{array}$$

induces an isomorphism on the π_k at any basepoint. Suppose the statement is proven for bounded hypercoverings, then the right vertical induces an isomorphism on π_k because $\text{cosk}_{k+1}^A B_\bullet$ is bounded. Hence the last map $\text{hocolim}_{\Delta^{op}} B_\bullet \rightarrow A$ induces an isomorphism on π_k at any basepoint hence is a weak equivalence because k is arbitrary.

Lemma 4.14. *Let $C_\bullet \rightarrow D_\bullet$ be a map of simplicial spaces which induces an isomorphisms on $(k+1)$ -skeleton. Then the map*

$$\pi_i \text{hocolim}_{\Delta^{op}} C_\bullet \rightarrow \pi_i \text{hocolim}_{\Delta^{op}} D_\bullet$$

is an isomorphism for every $0 \leq i \leq k$ and any basepoint.

To prove this, we reduce to the case of simplicial simplicial sets using the singular functor, because for simplicial simplicial sets the homotopy colimit is weakly equivalent to the diagonal. We denote by

$$S\text{Set} \xrightleftharpoons[S]{Re} \text{Top}$$

the standard adjunction with right adjoint the singular functor S . It induces an adjunction on the level of simplicial objects just by taking these functors levelwise. The counit map $Re \circ SC_\bullet \rightarrow C_\bullet$ is a levelwise weak equivalence in $\text{Top}^{\Delta^{op}}$, hence the induced map

$$\text{hocolim}_{\Delta^{op}} Re \circ SC_\bullet \rightarrow \text{hocolim}_{\Delta^{op}} C_\bullet$$

is a weak equivalence. But composing with the canonical weak equivalence $hocolim_{\Delta^{op}} Re \circ SC_{\bullet} \simeq Re(hocolim_{\Delta^{op}} Sing C_{\bullet})$, we get a weak equivalence $Re(hocolim_{\Delta^{op}} SC_{\bullet}) \simeq hocolim_{\Delta^{op}} C_{\bullet}$. Then for every $i \geq 0$ we get a canonical isomorphisms

$$\pi_i(hocolim_{\Delta^{op}} SC_{\bullet}) \simeq \pi_i Re(hocolim_{\Delta^{op}} SC_{\bullet}) \simeq \pi_i hocolim_{\Delta^{op}} C_{\bullet}$$

at every basepoint. Therefore it suffices to prove the claim for $C_{\bullet} \rightarrow D_{\bullet}$ a map in $SSet^{\Delta^{op}}$. But in that case there is canonical weak equivalence $hocolim_{\Delta^{op}} C_{\bullet} \simeq dC_{\bullet}$ in $SSet$ where $d : SSet^{\Delta^{op}} \rightarrow SSet$ is the diagonal functor. Then if $C_{\bullet} \rightarrow D_{\bullet}$ is an isomorphism on $(k+1)$ -skeleton, it is straightforward that $\pi_i dC_{\bullet} \rightarrow \pi_i dD_{\bullet}$ is an isomorphism for every $0 \leq i \leq k$. This finishes the proof of 4.14.

Back to the proof of 4.13, we will then proceed by induction on the dimension of the hypercovering, reducing the proof to the dimension 0 case.

Let $n \geq 0$ be an integer. Suppose we have proven the statement for hypercoverings of dimension $\leq n$ and let $B_{\bullet} \rightarrow A$ be a bounded proper hypercovering of dimension $n+1$. Consider the unit map $B_{\bullet} \rightarrow \text{cosk}_n^A B_{\bullet} =: C_{\bullet}$. Then C_{\bullet} is bounded of dimension $\leq n$. Consider the bisimplicial space which is the nerve of the map $B_{\bullet} \rightarrow C_{\bullet}$

$$E_{\bullet\bullet} := (B_{\bullet} \rightrightarrows B_{\bullet} \times_{C_{\bullet}} B_{\bullet} \rightrightarrows B_{\bullet} \times_{C_{\bullet}} B_{\bullet} \times_{C_{\bullet}} B_{\bullet} \cdots).$$

Considering C_{\bullet} as constant in one simplicial direction, we have a map $E_{\bullet\bullet} \rightarrow C_{\bullet}$. The k th row of $E_{\bullet\bullet} \rightarrow C_{\bullet}$ is the nerve of the map $B_k \rightarrow C_k$. Consider the diagonal $D_{\bullet} := dE_{\bullet\bullet}$. Then standard homotopy theory (see e.g. [16]) proves that $hocolim_{\Delta^{op}} D_{\bullet}$ is weakly equivalent to the space obtained by taking the homotopy colimit of each rows of $E_{\bullet\bullet}$, and then taking the homotopy colimit of the resulting simplicial space. But by induction hypothesis, the k th row being a dimension 0 hypercovering of C_k , its homotopy colimit is weakly equivalent to C_k . The resulting simplicial object is C_{\bullet} , which is of dimension $\leq n$, so by induction hypothesis $hocolim_{\Delta^{op}} C_{\bullet} \simeq A$. Hence we prove that $hocolim_{\Delta^{op}} D_{\bullet} \simeq A$.

Now we prove that B_{\bullet} is a retract of D_{\bullet} over A , hence that $hocolim_{\Delta^{op}} D_{\bullet} \simeq hocolim_{\Delta^{op}} B_{\bullet} \simeq A$. There is a natural map $B_{\bullet} \rightarrow D_{\bullet}$ given by the horizontal degeneracy $E_{0,k} \rightarrow E_{k,k}$. Then we need a map $D_{\bullet} \rightarrow B_{\bullet}$. It is sufficient to find a map $sk_{n+1}^A D_{\bullet} \rightarrow sk_{n+1}^A B_{\bullet}$ because then the adjoint map $D_{\bullet} \rightarrow \text{cosk}_{n+1}^A sk_{n+1}^A B_{\bullet} \simeq B_{\bullet}$ is the wanted map. Notice that because of the definition of C_{\bullet} the map $B_k \rightarrow C_k$ is an isomorphism for $k = 0, \dots, n$ and the map $sk_n^A B_{\bullet} \rightarrow sk_n^A D_{\bullet}$ is an isomorphism. Let $[0] \rightarrow [n+1]$ be any coface map, giving a face map $E_{n+1,n+1} \rightarrow E_{0,n+1}$ which gives the wanted map $sk_{n+1}^A D_{\bullet} \rightarrow sk_{n+1}^A B_{\bullet}$. One can check that $B_{\bullet} \rightarrow D_{\bullet} \rightarrow B_{\bullet}$ is the identity which proves our claim.

It remains to prove the statement for a dimension 0 proper hypercovering $\pi : B_{\bullet} \rightarrow A$ with spaces satisfying the assumptions of 4.13. Such a hypercovering is the nerve of a proper surjective map $B_0 \rightarrow A$. Therefore $B_n \simeq B_0 \times_A \cdots \times_A B_0$, $n+1$ times. We will use a proper base change argument. For this we will study simplicial presheaves on the simplicial space B_{\bullet} and their behavior with respect to π . Deligne defined in [5] a notion of sheaves on a simplicial space, constructing a site out of a simplicial space and taking sheaves on it. His construction can be directly use for simplicial presheaves. Indeed let \tilde{B}_{\bullet} be the category with objects the pairs $([n], V)$ with $[n] \in \Delta$ and $V \subseteq B_n$ an open subset. A morphism between $([n], V)$ and $([m], V')$ is the data of a morphism $a : [n] \rightarrow [m]$ in Δ and a continuous map $V' \rightarrow V$ such that the square

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ B_m & \xrightarrow{B(a)} & B_n \end{array}$$

commute. Composition and identities are defined in the obvious way, and satisfy all the required conditions. The category \tilde{B}_{\bullet} is naturally endowed with the open covering topology induced by the topology of each B_n and Δ is considered as discrete. Then we can consider the category $SPR(B_{\bullet})$ of simplicial

presheaves on the site \tilde{B}_\bullet . An object F in this category is equivalent to the data of simplicial presheaves F_n on B_n for every $n \geq 0$, and for every map $a : [n] \rightarrow [m]$ in Δ , a map of presheaves $u_a : F_n \rightarrow B(a)_* F_m$, such that $u_{id_{[n]}} = id_{F_n}$ and for every $a : [n] \rightarrow [m]$ and $b : [m] \rightarrow [k]$ in Δ , we have $u_{ba} = B(a)_* u_b u_a$.

If X is any space considered as a constant simplicial space, then \tilde{X} is nothing but the site of opens of X . The map of simplicial spaces $\pi : B_\bullet \rightarrow A$ gives a map of sites still denoted by $\pi : \tilde{B}_\bullet \rightarrow \tilde{A}$. Consider the diagram of categories

$$\begin{array}{ccc} \tilde{B}_\bullet & \xrightarrow{\pi} & \tilde{A} \\ & \searrow q & \swarrow p \\ & * & \end{array}$$

where $*$ is the punctual category. Then taking simplicial presheaves we get a set of adjoint functors

$$\begin{array}{ccc} SPr(B_\bullet) & \xrightleftharpoons[\pi^{-1}]{\pi_*} & SPr(A) \\ \swarrow q_* & & \searrow p_* \\ SSet & \xrightleftharpoons[cst]{cst} & SSet \end{array}$$

where $cst(K)$ is the constant simplicial presheaf with value K for any $K \in SSet$. The functors p_* and q_* are also famous under the name of global sections and are the right adjoints to cst . We endow $SPr(B_\bullet)$ and $SPr(A)$ with the "local" (see definition) Bousfield localization of the injective model structure (what is really important is the weak equivalences which are local equivalences, but we will need below to consider a homotopy limit, that is why we need the injective one). Then the functors π_* , p_* and q_* are right Quillen. For any $K \in SSet$ we have $\pi^{-1} \circ cst(K) \simeq cst(K)$ and there are isomorphisms $\mathbb{L}\pi^{-1} \simeq \pi^{-1}$ and $\mathbb{L}cst \simeq cst$. Therefore we have a canonical isomorphism $\mathbb{R}p_* \mathbb{R}\pi_* \simeq \mathbb{R}q_*$.

A direct consequence of Toën's result [37, Theorem 2.13] is that for any constant simplicial presheaf $K \in SPr(A)$ we have a canonical isomorphism $\mathbb{R}p_*(K) \simeq \mathbb{R}Map(SA, K)$ in $Ho(SSet)$. This uses the fact that our space A has the homotopy type of a CW complex. Next we want to calculate the derived global sections $\mathbb{R}q_*(K)$ of a constant simplicial presheaf K on B_\bullet . The site \tilde{B}_\bullet is naturally endowed with a functor $\alpha : \tilde{B}_\bullet \rightarrow \Delta$ where Δ is considered as a discrete site. The functor α is just the projection $\alpha([n], V) = [n]$. We denote by $\beta : \Delta \rightarrow *$ the unique functor. We have induced functors on simplicial presheaves

$$\begin{array}{ccc} SPr(B_\bullet) & \xrightarrow{\alpha_*} & SPr(\Delta) \\ & \searrow q_* & \downarrow \beta_* \\ & & SSet \end{array}$$

These functors are right Quillen ($SPr(\Delta)$ is also endowed with the injective model structure). The derived functor $\mathbb{R}\beta_*$ is then isomorphic to $holim_\Delta$. For any constant simplicial presheaf $K \in SPr(B_\bullet)$, we have $\mathbb{R}\alpha_*(K) = (\mathbb{R}q_{n*}K)_{n \geq 0}$ where $q_n : \tilde{B}_n \rightarrow *$. Using [37, Theorem 2.13] (all spaces B_n having the homotopy type of a CW), we obtain an isomorphism $\mathbb{R}\alpha_*(K) \simeq (\mathbb{R}Map(SB_n, K))_{n \geq 0}$ in $Ho(SPr(\Delta))$. Therefore we have a canonical isomorphism in $Ho(SSet)$

$$\mathbb{R}q_*(K) \simeq holim_{\Delta^{op}} \mathbb{R}Map(SB_\bullet, K) \simeq \mathbb{R}Map(hocolim_{\Delta^{op}} SB_\bullet, K).$$

Suppose for the time being that we have the following lemma at our disposal.

Lemma 4.15. *Let $K \in SPr(B_\bullet)$ be a constant simplicial presheaf, with K being a truncated simplicial set (i.e. isomorphic to one of its skeleton). Then the unit map*

$$K \longrightarrow \mathbb{R}\pi_* \pi^{-1}(K) \simeq \mathbb{R}\pi_*(K)$$

is an isomorphism in $Ho(SPr(A))$ (where K denotes the same constant presheaf on A).

We will give a proof below. Applying the isomorphism $\mathbb{R}p_*\mathbb{R}\pi_* \simeq \mathbb{R}q_*$ to a truncated constant simplicial presheaf K on B_\bullet , we obtain a canonical isomorphism for every truncated simplicial set K

$$\mathbb{R}\mathrm{Map}(SA, K) \simeq \mathbb{R}\mathrm{Map}(\mathrm{hocolim}_{\Delta^{op}} SB_\bullet, K).$$

This implies that the map $\mathrm{hocolim}_{\Delta^{op}} SB_\bullet \rightarrow SA$ is an isomorphism in $Ho(SSet)$. By taking the Quillen equivalence Re and the fact that Re commutes with homotopy colimits, this implies that the map $\mathrm{hocolim}_{\Delta^{op}} B_\bullet \rightarrow A$ is an isomorphism in $Ho(Top)$, proving our claim.

To sum up, it only remains to prove 4.15.

To prove 4.15 it suffices to prove that the unit map is an isomorphism on the stalk of any point $a \in A$. We have a Cartesian square of simplicial spaces

$$\begin{array}{ccc} B_\bullet^a & \xrightarrow{\pi^a} & * \\ \phi \downarrow & & \downarrow a \\ B_\bullet & \xrightarrow{\pi} & A \end{array}$$

We claim that the base change theorem holds for this square and for a truncated constant simplicial presheaf. This is the following.

Lemma 4.16. *For any truncated constant simplicial presheaf $K \in SPr(B_\bullet)$, the canonical map*

$$a^{-1}\mathbb{R}\pi_*(K) \rightarrow \mathbb{R}\pi_*^a \phi^{-1}(K) \simeq \mathbb{R}\pi_*^a(K)$$

is an isomorphism in $Ho(SSet)$.

Once the lemma is proven, it will just remain to prove that the map $K \rightarrow \mathbb{R}\pi_*^a(K)$ is an isomorphism in $Ho(SSet)$, which is exactly the statement of 4.13 for $A = *$ and B_\bullet a dimension 0 hypercovering. Indeed we have an isomorphism $\mathbb{R}\pi_*^a(K) \simeq \mathrm{holim}_{\Delta^{op}} \mathbb{R}\mathrm{Map}(SB_\bullet^a, K)$.

To prove 4.16, we will calculate the stalk $a^{-1}\mathbb{R}\pi_*(K)$ and relate it to $\mathbb{R}\pi_*^a(K)$. For any open subset $U \subseteq A$ we have a Cartesian square of simplicial spaces

$$\begin{array}{ccc} B_\bullet^U & \xrightarrow{\pi^U} & U \\ \phi^U \downarrow & & \downarrow i \\ B_\bullet & \xrightarrow{\pi} & A \end{array}$$

Then we have isomorphisms $\mathbb{R}\Gamma(U, \mathbb{R}\pi_*^U(K)) \simeq \mathbb{R}\Gamma(B_\bullet^U, K) \simeq \mathrm{holim}_{\Delta^{op}} \mathbb{R}\mathrm{Map}(SB_\bullet^U, K)$ in $Ho(SSet)$. The stalk $a^{-1}\mathbb{R}\pi_*(K)$ is isomorphic to usual filtered colimit

$$\mathrm{colim}_{a \in U \subseteq A} \mathbb{R}\Gamma(U, \mathbb{R}\pi_*^U(K)) \simeq \mathrm{colim}_{a \in U \subseteq A} \mathrm{holim}_{\Delta^{op}} \mathbb{R}\mathrm{Map}(SB_\bullet^U, K).$$

Now we use the assumption that K is truncated to deduce the fact this homotopy limit is isomorphic to a finite homotopy limit. Indeed if K is n -truncated, then the simplicial set $\mathbb{R}\mathrm{Map}(SB_\bullet^U, K)$ is also n -truncated and one can calculate this homotopy limit by restricting to the subcategory of Δ^{op} given by simplexes of dimension $\leq n+1$. Then we can make this filtered colimit and this finite homotopy limit commute to get

$$a^{-1}\mathbb{R}\pi_*(K) \simeq \mathrm{holim}_{\Delta^{op}} \mathrm{colim}_{a \in U \subseteq A} \mathbb{R}\mathrm{Map}(SB_\bullet^U, K).$$

Now we wish to have for all $n \geq 0$ an isomorphism $\mathrm{colim}_{a \in U \subseteq A} \mathbb{R}\mathrm{Map}(SB_n^U, K) \simeq \mathbb{R}\mathrm{Map}(SB_n^a, K)$. To do so we apply Lurie's proper base change Theorem [26, Corollary 7.3.1.18] to the Cartesian square of

locally compact Hausdorff spaces

$$\begin{array}{ccc} B_n^a & \xrightarrow{\pi_n^a} & * \\ \phi_n \downarrow & & \downarrow a \\ B_n & \xrightarrow{\pi_n} & A \end{array}$$

We can apply this result here because our simplicial presheaf K is truncated so that according to [26, Corollary 7.2.1.12], K satisfies hyperdescent if and only if K satisfies ordinary descent. We obtain an isomorphism $a^{-1}\mathbb{R}\pi_{n*}(K) \simeq \mathbb{R}\pi_{n*}^a(K)$. But with the same argument as before $a^{-1}\mathbb{R}\pi_{n*}(K) \simeq \text{colim}_{a \in U \subseteq A} \mathbb{R}\text{Map}(SB_n^U, K)$, which proves the desired isomorphism. This finishes the proof of Lemma 4.16.

Back to the proof of 4.15, it remains to prove that the map $K \rightarrow \mathbb{R}\pi_*^a(K)$ is an isomorphism in $Ho(SSet)$ for all truncated K . In view of what has been said, it is equivalent to the statement that $\text{hocolim}_{\Delta^{op}} B_\bullet^a \rightarrow *$ is a weak equivalence of spaces. It is treated by the following lemma.

Lemma 4.17. *Let X be any non empty topological space (resp. a non empty simplicial set). Then the nerve $X_\bullet \rightarrow *$ of the map $p : X \rightarrow *$ induces a weak equivalence $\text{hocolim}_{\Delta^{op}} X_\bullet \rightarrow *$ in Top (resp. in $SSet$).*

The statement in $SSet$ implies the statement in Top . Indeed if $X \in Top$ we saw in the proof of 4.14 that $\text{hocolim}_{\Delta^{op}} X_\bullet$ and $\text{hocolim}_{\Delta^{op}} SX_\bullet$ have the same homotopy groups.

Let $X \in SSet$. We prove that the map $X_\bullet \rightarrow *$ is a simplicial homotopy equivalence in $SSet^{\Delta^{op}}$. Let $x : * \rightarrow X$ be a point. Then it suffices to find a homotopy $h : \Delta^1 \times X_\bullet \rightarrow X_\bullet$ between id_{X_\bullet} and xp . We define $h_n : \Delta([n], [1]) \times X_n \rightarrow X_n$ by the following formula. Let $a : [n] \rightarrow [1]$ be a map in Δ , it is essentially given by an integer $0 \leq m \leq n$. We set $h_n(a, (x_0, \dots, x_n)) = (x_0, \dots, x_m, x, \dots, x)$. We then have a homotopy which verifies $h(0, -) = id_{X_\bullet}$ and $h(1, -) = xp$.

Recall the realization functor

$$|-| : SSet^{\Delta^{op}} \rightarrow SSet$$

defined by the standard formula

$$|Y_\bullet| := \text{coeq}(\bigsqcup_{n \in \Delta} \Delta^n \times Y_n \rightrightarrows \bigsqcup_{p \rightarrow q \in \Delta} \Delta^p \times Y_q).$$

It has the property that it sends simplicial homotopy equivalences to simplicial homotopy equivalences. This implies that $|X_\bullet|$ is contractible. Now we use the isomorphism $\text{hocolim}_{\Delta^{op}} X_\bullet \simeq |X_\bullet|$ in $Ho(SSet)$ (see [16]) to conclude that $\text{hocolim}_{\Delta^{op}} X_\bullet$ is contractible.

Now the proof of 4.13 is complete. \square

4.5 Topological Chern character

Recall from Definition 3.5 that for every $T \in dgCat/\mathbb{C}$ we have a $\underline{\mathbf{K}}(\mathbb{C})$ -linear Chern character

$$ch_T : \underline{\mathbf{K}}(T) \rightarrow \underline{\mathbf{HC}}^-(T).$$

We compose this map with the natural map $\underline{\mathbf{HC}}^-(T) \rightarrow \underline{\mathbf{HP}}(T)$ to obtain a $\underline{\mathbf{K}}(\mathbb{C})$ -linear map

$$\underline{\mathbf{K}}(T) \rightarrow \underline{\mathbf{HP}}(T).$$

Because the functor $|-|$ commutes with smash products, the induced map

$$\mathbf{K}^{\text{st}}(T) \simeq \mathbb{L}|\underline{\mathbf{K}}(T)| \rightarrow \mathbb{L}|\underline{\mathbf{HP}}(T)|$$

is a map of **bu**-modules. Recall that the object $\underline{\mathbf{HP}}(T)$ is the presheaf of spectra $A \mapsto \mathbf{HP}(T \otimes_{\mathbb{C}}^{\mathbb{L}} A)$. Now the idea is to use a Künneth type formula for periodic cyclic homology in order to calculate $\mathbf{HP}(T \otimes_{\mathbb{C}}^{\mathbb{L}} A)$ for any smooth commutative algebra A . Indeed Kassel's Theorem [21, Thm 2.3] together with [21, Prop 2.4] implies that for any smooth finite type commutative \mathbb{C} -algebra A , there exists a natural map

$$\mathbf{HP}(T) \xrightarrow{h} \mathbf{HP}(T \otimes_{\mathbb{C}}^{\mathbb{L}} A),$$

which is a weak equivalence. Therefore the map $\mathbf{HP}(T) \xrightarrow{h} \mathbf{HP}(T \otimes_{\mathbb{C}}^{\mathbb{L}} A) \longrightarrow \mathbf{HP}(T, -)$ is an étale-proper local equivalence (see Definition 4.11). By Proposition 4.12 and Remark 2.22 we obtain an isomorphism in $Ho(Sp^{\Sigma})$,

$$\mathbb{L}|\underline{\mathbf{HP}}(T)| \simeq \mathbf{HP}(T) \xrightarrow{h} \mathbb{L}|A \mapsto \mathbf{HP}(A)|.$$

In order to obtain a (functorial) map $\mathbf{K}^{\text{st}} \longrightarrow \mathbf{HP}$, defined in $Ho(Fun(dgCat/\mathbb{C}, Sp^{\Sigma}))$, it suffices to choose a map

$$\mathbb{L}|A \mapsto \mathbf{HP}(A)| \longrightarrow H\mathbb{C}[u, u^{-1}]$$

in $Ho(Sp^{\Sigma})$. Then by adjunction, it suffices to choose a map in $Ho(Sp^{\Sigma}(\mathbb{C})^{\text{ét}})$ (see 4.11 for definitions) :

$$\mathbf{HP} \longrightarrow (H\mathbb{C}[u, u^{-1}])_B.$$

The presheaf of spectra $(H\mathbb{C}[u, u^{-1}])_B$ is by definition (2-periodic) Betti cohomology, i.e. the presheaf

$$A \mapsto \mathbb{R}\underline{\mathbf{Hom}}_{Ho(Sp^{\Sigma})}(|A|, H\mathbb{C}[u, u^{-1}]) \simeq \mathbb{R}\underline{\mathbf{Hom}}_{Ho(Sp^{\Sigma})}(|A|, H\mathbb{C}) \xrightarrow{h} H\mathbb{C}[u, u^{-1}].$$

We denote the actual Betti cohomology $\mathbb{R}\underline{\mathbf{Hom}}_{Ho(Sp^{\Sigma})}(|-|, H\mathbb{C})$ by H_B and 2-periodic Betti cohomology by $H_B[u, u^{-1}]$. Thus we are looking for a map

$$\mathbf{HP} \longrightarrow H_B[u, u^{-1}]$$

in $Ho(Sp^{\Sigma}(\mathbb{C})^{\text{ét}})$. For this we will use the classical antisymmetrisation map

$$\mathbf{HP} \longrightarrow H_{\text{DR}}^{\text{naive}}$$

which goes from periodic cyclic homology to naive De Rham cohomology, i.e. the spectrum which calculates the hypercohomology of the complex of sheaves of Khäler differential forms. The inclusion of algebraic forms into complex analytic forms induces a map

$$H_{\text{DR}}^{\text{naive}} \longrightarrow H_{\text{DR}}^{\text{naive, an}},$$

where $H_{\text{DR}}^{\text{naive, an}}$ is the (presheaf of spectra of) complex analytic De Rham cohomology. The morphism from the constant sheaf \mathbb{C} to the complex of sheaves of complex analytic forms gives a map

$$H_{\text{DR}}^{\text{naive, an}} \longleftarrow H_B,$$

which is a weak equivalence on smooth $A \in \mathcal{CAlg}/\mathbb{C}$, and thus an étale-proper local equivalence. We end up with a zig-zag

$$\mathbf{HP} \longrightarrow H_{\text{DR}}^{\text{naive}} \longrightarrow H_{\text{DR}}^{\text{naive, an}} \xleftarrow{\sim} H_B$$

which gives a well defined map $\mathbf{HP} \longrightarrow H_B[u, u^{-1}]$ in $Ho(Sp^{\Sigma}(\mathbb{C})^{\text{ét}})$. In fact this map is essentially the classical "period map" from De Rham cohomology to Betti cohomology, and the morphisms $\mathbf{HP} \longrightarrow$

$H_{\text{DR}}^{\text{naive}} \longrightarrow H_{\text{DR}}^{\text{naive,an}}$ are also an equivalence on smooth $A \in \text{Calg}/\mathbb{C}$ (by the HKR Theorem and Grothendieck comparison Theorem respectively). Thus we obtain in this way the "semi-topological" Chern character

$$ch^{\text{st}} : \mathbf{K}^{\text{st}} \longrightarrow \text{HP}.$$

There is also a connective version of it $ch^{\text{cst}} : \tilde{\mathbf{K}}^{\text{st}} \longrightarrow \text{HP}$.

Now if $T = \mathbb{C}$, we have the map $ch_{\mathbb{C}}^{\text{st}} : \mathbf{K}^{\text{st}}(\mathbb{C}) = |\underline{\mathbf{K}}(\mathbb{C})| \longrightarrow \text{HP}(\mathbb{C}) \simeq H\mathbb{C}[u, u^{-1}]$. Then if we look on π_2 we have $ch_{\mathbb{C}}^{\text{st}}(\beta) = u$. This comes from the fact that the map $ch_{\mathbb{C}}^{\text{st}}$ corresponds by adjunction to the map

$$ch_{\mathbb{C}} : \underline{\mathbf{K}}(\mathbb{C}) \longrightarrow H_{\mathbb{B}}[u, u^{-1}]$$

in $Ho(Sp^{\Sigma}(\mathbb{C}))$, which is actually the usual algebraic Chern character (see Remark 3.7), and from the classical fact that β corresponds via the geometric realization to the first Chern class of the universal line bundle $\mathcal{O}(-1)$ in the cohomology of \mathbf{P}^1 . Therefore by the Definition 4.5 of topological K-theory and the universal property of localization we obtain a map in $Ho(Fun(dgCat/\mathbb{C}, Sp^{\Sigma}))$

$$ch^{\text{top}} : \mathbf{K}^{\text{top}} \longrightarrow \text{HP}.$$

We have also a connective version $ch^{\text{ctop}} : \tilde{\mathbf{K}}^{\text{top}} \longrightarrow \text{HP}$.

The algebraic K-theory is related to the topological one by a natural map. Let $T \in dgCat/\mathbb{C}$ and consider the unit of the adjunction $(|-|, (-)_B)$

$$\underline{\mathbf{K}}(T) \longrightarrow (|\underline{\mathbf{K}}(T)|)_B = \mathbb{R}\underline{\text{Hom}}(|-|, |\underline{\mathbf{K}}(T)|) \simeq \mathbb{R}\underline{\text{Hom}}(|-|, \mathbf{K}^{\text{st}}(T)).$$

Then take global sections, i.e. the value on $\text{Spec}(\mathbb{C})$ to obtain a map

$$\mathbf{K}(T) = \underline{\mathbf{K}}(T)(\mathbb{C}) \longrightarrow \mathbb{R}\underline{\text{Hom}}(|\mathbb{C}|, \mathbf{K}^{\text{st}}(T)) \simeq \mathbf{K}^{\text{st}}(T).$$

The following theorem follows from the definition of the topological Chern character.

Theorem 4.18. *The algebraic Chern character map $\mathbf{K}(T) \longrightarrow \text{HC}^-(T)$ factorizes through topological K-theory providing a commutative square in $Ho(Sp^{\Sigma})$,*

$$\begin{array}{ccc} \mathbf{K}(T) & \xrightarrow{ch} & \text{HC}^-(T) \\ \downarrow & & \downarrow \\ \mathbf{K}^{\text{top}}(T) & \xrightarrow{ch^{\text{top}}} & \text{HP}(T). \end{array}$$

Remark 4.19. This results enables us to define Deligne cohomology of dg-categories by the formula $H_{\text{Del}}(T) := \mathbf{K}^{\text{top}}(T) \overset{h}{\times}_{\text{HP}(T)} \text{HC}^-(T)$ and to observe the existence of a Chern character $\mathbf{K}(T) \longrightarrow H_{\text{Del}}(T)$.

5 Conjectures

Recall from the notion of smooth dg-category and proper dg-category. A dg-category T is proper if its complexes of morphisms are perfect complexes, and if the triangulated category $[\widehat{T}]$ has a compact generator. A dg-category T is smooth if the $T^{\text{op}} \otimes^{\mathbb{L}} T$ -dg-module $(x, y) \mapsto T(x, y)$ is perfect.

The following conjecture which has been communicated to me by Bertrand Toën is a generalization of 4.3 to dg-categories. It seems closely related to the Hodge to De-Rham degeneration.

Conjecture 5.1. *For any dg-category T smooth and proper over \mathbb{C} , the map $\tilde{\mathbf{K}}^{\text{st}}(T) \longrightarrow \mathbf{K}^{\text{st}}(T)$ is a weak equivalence.*

The following conjecture is a reformulation of conjectures made in [22, 2.2.6] and [20, 8, Conj 8.6]. Its validity would implies the existence of the Betti part of a non-commutative Hodge structure on $\mathrm{HP}(T)$.

Conjecture 5.2. *For any dg-category T smooth and proper over \mathbb{C} , the map*

$$ch^{top} \wedge_{\mathrm{BU}} H\mathbb{C} : \mathbf{K}^{\mathrm{top}}(T) \wedge_{\mathrm{BU}} H\mathbb{C} \longrightarrow \mathrm{HP}(T)$$

is a weak equivalence.

An important result of [41] says that a smooth and proper dg-category is of finite type, hence have a compact generator and is quasi-equivalent to a smooth and proper dg-algebra. Because topological K-theory and periodic cyclic homology are invariant under quasi-equivalences, we don't loose any generality in stating the two conjectures with smooth and proper dg-algebras.

Modulo a comparison statement between our definition and Friedlander–Walker's definition of semi-topological K-theory, the two conjectures above are true for $T = L_{pe}(X)$ the dg-category of perfect complexes of quasi-coherent sheaves on a smooth and proper complex algebraic variety X . Hence they are also true for direct factors of such a dg-category. We also expect them to be true for an associative \mathbb{C} -algebra of finite dimension as a \mathbb{C} -vector space.

References

- [1] A. Blanc, *K-théorie topologique des espaces non-commutatifs*, Ph.D. thesis, Université Montpellier 2, in preparation.
- [2] A. Bousfield and E. Friedlander, *Homotopy theory of gamma-spaces, spectra, and bisimplicial sets*, Geometric applications of homotopy theory II (1978), 80–130.
- [3] D.C. Cisinski and G. Tabuada, *Symmetric monoidal structure on non-commutative motives*, Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology **1**, no. 1, 1–68.
- [4] ———, *Non-connective K-theory via universal invariants*, Compositio Mathematica **147** (2011), no. 04, 1281–1320.
- [5] P. Deligne, *Théorie de Hodge III*, Publications Mathématiques de l'IHÉS **44** (1974), no. 1, 5–77.
- [6] D. Dugger, *Universal homotopy theories*, Advances in Mathematics **164** (2001), no. 1, 144–176.
- [7] D. Dugger, S. Hollander, and D.C. Isaksen, *Hypercovers and simplicial presheaves*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 136, Cambridge Univ Press, 2004, pp. 9–51.
- [8] D. Dugger and D.C. Isaksen, *Hypercovers in topology*, Arxiv preprint math/0111287 (2001).
- [9] T. Dyckerhoff, *Compact generators in categories of matrix factorizations*, Duke Mathematical Journal **159** (2011), no. 2, 223–274.
- [10] D.S. Freed, *Remarks on Chern-Simons theory*, Bull. Amer. Math. Soc **46** (2009), 221–254.
- [11] E. Friedlander and M. Walker, *Comparing K-theories for complex varieties*, American Journal of Mathematics (2001), 779–810.
- [12] E.M. Friedlander and B. Mazur, *Filtrations on the homology of algebraic varieties*, no. 529, American Mathematical Soc., 1994.

- [13] E. Getzler, *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology*, Israel Math. Conf. Proc, vol. 7, 1993, pp. 65–78.
- [14] H. Gillet, *Riemann-roch theorems for higher algebraic K-theory*, Advances in Mathematics **40** (1981), no. 3, 203–289.
- [15] A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondamental: Séminaire de géométrie algébrique du bois marie 1960-61 SGA I*, vol. 224, Springer, 1971.
- [16] P.S. Hirschhorn, *Model categories and their localizations*, AMS Bookstore, 2009.
- [17] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.
- [18] M. Hovey, B. Shipley, and J. Smith, *Symmetric spectra*, Journal of the AMS **13** (2000), no. 1, 149–208.
- [19] D. Kaledin, *Non-commutative Hodge-to-de Rham degeneration via the method of Deligne-Illusie*, Pure and Applied Mathematics Quarterly **4**, no. 3.
- [20] ———, *Motivic structures in non-commutative geometry*, Proceeding of the ICM, vol. 901, 2010, pp. 461–496.
- [21] C. Kassel, *Cyclic homology, comodules, and mixed complexes*, Journal of Algebra **107** (1987), no. 1, 195–216.
- [22] L. Katzarkov, M. Kontsevich, and T. Pantev, *Hodge theoretic aspects of mirror symmetry*, Arxiv preprint arxiv:0806.0107 (2008).
- [23] B. Keller, *On the cyclic homology of exact categories*, Journal of Pure and Applied Algebra **136** (1999), no. 1, 1–56.
- [24] ———, *On differential graded categories*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.
- [25] M. Kontsevich and Y. Soibelman, *Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. i*, Arxiv preprint math/0606241 (2006).
- [26] J. Lurie, *Higher topos theory*, (2009).
- [27] F. Morel and V. Voevodsky, *A1-homotopy theory of schemes*, Publications Mathématiques de l’IHES **90** (1999), no. 1, 45–143.
- [28] M. Schlichting, *Negative K-theory of derived categories*, Mathematische Zeitschrift **253** (2006), no. 1, 97–134.
- [29] S. Schwede, *An untitled book project about symmetric spectra*, available on the author’s web page.
- [30] S. Schwede and B.E. Shipley, *Algebras and modules in monoidal model categories*, Proceedings of the London Mathematical Society **80** (2000), no. 2, 491.
- [31] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), no. 3, 293–312.
- [32] D. Shklyarov, *Hirzebruch-Riemann-Roch theorem for dg algebras*, arXiv preprint arXiv:0710.1937 (2007).

- [33] G. Tabuada, *Théorie homotopique des dg-catégories*, Ph.D. thesis, arXiv:0710.4303v1 [math.KT], 2007.
- [34] ———, *Higher K-theory via universal invariants*, Duke Math. J. **145** (2008), no. 1, 121–206.
- [35] R. W. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435.
- [36] R.W. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Sci. École Norm. Sup.(4) **18** (1985), no. 3, 437–552.
- [37] B. Toën, *Vers une interprétation galoisienne de la théorie de l'homotopie*, Cahiers de topologie et géométrie différentielle catégoriques **43** (2002), no. 4, 257–312.
- [38] ———, *Derived Hall algebras*, Duke Mathematical Journal **135** (2006), no. 3, 587–615.
- [39] ———, *The homotopy theory of dg-categories and derived Morita theory*, Invent. Math. **167** (2007), no. 3, 615–667.
- [40] ———, *Lectures on dg-categories*, Topics in Algebraic and Topological K-theory, Lectures notes in math., vol. 2008, Springer, 2010.
- [41] B. Toën and M. Vaquié, *Moduli of objects in dg-categories*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 3, 387–444.
- [42] B. Tsygan, *On the gauss-manin connection in cyclic homology*, Methods Funct. Anal. Topology **13** (2007), no. 1, 83–94.
- [43] F. Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419.